6th May, 2021

MAT4520 - Spring 2021

Mandatory assignment 1 of 1

Submission deadline

Thursday 22th April 2021, 14:30 in Canvas.

Instructions

The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number. Submission in LaTeXis preferred.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. Students who fail the assignment, but have made a genuine effort at solving the exercises, are given a second attempt at revising their answers. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) well before the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

Problem 1. Let V be a 3-dimensional vector space over $\mathbb{R}$ with a basis $\mathcal{B} = \{e_1, e_2, e_3\}$. We will consider the graded algebra $\oplus_{k=0}^{\infty} A_k(V)$ (here $A_0(V) = \mathbb{R}$, and $A_k(V) = 0$ for $k > 3$). Let $\{e_1^*, e_2^*, e_3^*\}$ be the dual basis to $\mathcal{B}$, and recall that

$$\{e_1^* \wedge e_2^*, e_1^* \wedge e_3^*, e_2^* \wedge e_3^*\} \text{ and } \{e_1^* \wedge e_2^* \wedge e_3^*\}$$

are bases for $A_2(V)$ and $A_3(V)$ respectively.

We will now define a map $\star : \oplus_{k=0}^{3} A_k(V) \to \oplus_{k=0}^{3} A_k(V)$. Define $\star : A_1(V) \to A_2(V)$ by setting $\star e_1^* = e_2^* \wedge e_3^*, \star e_2^* = -e_1^* \wedge e_3^*$, $\star e_3^* = e_1^* \wedge e_2^*$, and extend by linearity. Define $\star : A_2(V) \to A_1(V)$ such that $\star^2 = \text{id}$. Define $\star : A_0(V) \to A_3(V)$ by setting $\star \lambda = \lambda e_1^* \wedge e_2^* \wedge e_3^*$ for $\lambda \in \mathbb{R}$, and define $\star : A_3(V) \to A_0(V)$ such that $\star^2 = \text{id}$. (This operator is called the *Hodge-$\star$* operator.) Finally, for each $k = 0, 1, 2, 3$, and $\alpha, \beta \in A_k(V)$, we define $\langle \alpha, \beta \rangle = \star(\alpha \wedge \star \beta)$.

(a) Show that for each k we have that $\langle \cdot, \cdot \rangle$ defines a symmetric bilinear form

$$\langle \cdot, \cdot \rangle : A_k(V) \times A_k(V) \to \mathbb{R}.$$

Solution: $k = 0$. We have $\langle \lambda, \mu \rangle = \star(\lambda \mu e_1^* \wedge e_2^* \wedge e_3^* = \lambda \mu)$.

$k = 1$. Write $\alpha = a_1 e_1^* + b_1 e_2^* + c_1 e_3^*$ and $\beta = a_2 e_1^* + b_2 e_2^* + c_2 e_3^*$. Then

$$\star \beta = a_2 e_2^* \wedge e_3^* - b_2 e_1^* \wedge e_3^* + c_2 e_2^* \wedge e_3^*,$$

and so

$$\begin{aligned} \alpha \wedge \star \beta &= a_1 e_1^* \wedge a_2 e_2^* \wedge e_3^* - b_1 e_2^* \wedge b_2 e_1^* \wedge e_3^* + c_1 e_3^* \wedge c_2 e_1^* \wedge e_2^* \\ &= (a_1 a_2 + b_1 b_2 + c_1 c_2) e_1^* \wedge e_2^* \wedge e_3^*, \end{aligned}$$

and so $\star(\alpha \wedge \star \beta) = a_1 a_2 + b_1 b_2 + c_1 c_2$.

$k = 2$. Similar to $k = 1$.

$k = 3$. Similar to $k = 0$.

(b) Show that for each k we have that $\langle \cdot, \cdot \rangle$ is an inner product on $A_k(V)$, and find an orthonormal basis for each $A_k(V)$.

Solution: By (a) we see that with respect to the given bases, these are just the Euclidean inner products on $\mathbb{R}$ and $\mathbb{R}^3$ respectively.

(c) For $\alpha, \beta \in A_1(V)$ we now define $[\alpha, \beta] = \star(\alpha \wedge \beta)$. Show that $[\cdot, \cdot]$ is an anticommutative bilinear form on $A_1(V)$ (it is in fact a Lie bracket).

Solution: Since $\wedge$ is anticommutative and bilinear and since $\star$ is linear, we get that $[\cdot, \cdot]$ anticommutative and bilinear .

(d) For $V = \mathbb{R}^3$ and $\mathcal{B}$ the standard basis on $\mathbb{R}^3$, express the relation between $[\cdot, \cdot]$ and the cross-product on $\mathbb{R}^3$.

Solution: Write $\alpha = a_1 e_1^* + b_1 e_2^* + c_1 e_3^*$ and $\beta = a_2 e_1^* + b_2 e_2^* + c_2 e_3^*$. Then

$$\begin{aligned}\alpha \wedge \beta &= a_1 b_2 e_1^* \wedge e_2^* + a_1 c_2 e_1^* \wedge e_3^* \\ &+ b_1 a_2 e_2^* \wedge e_1^* + b_1 c_2 e_2^* \wedge e_3^* \\ &+ c_1 a_2 e_3^* \wedge e_1^* + c_1 b_2 e_3^* \wedge e_2^* \\ &= (a_1 b_2 - b_1 a_2) e_1^* \wedge e_2^* + (a_1 c_2 - c_1 a_2) e_1^* \wedge e_3^* + (b_1 c_2 - c_1 b_2) e_2^* \wedge e_3^*.\end{aligned}$$

So

$$\star(\alpha \wedge \beta) = (a_1 b_2 - b_1 a_2) e_3^* - (a_1 c_2 - c_1 a_2) e_2^* + (b_1 c_2 - c_1 b_2) e_1^*.$$

So if $l : V \to V^*$ is defined by $l(e_j) = e_j^*$ we see that for $v, w \in V$ we have that $v \times w = l^{-1}[l(v), l(w)]$.

Problem 2. Let M be a smooth manifold, and let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas on M. Suppose that for any pair α, β with $U_{\beta\alpha} = U_\beta \cap U_\alpha$ nonempty, we have a smooth map $\phi_{\beta\alpha} : U_{\beta\alpha} \to GL(n, \mathbb{R})$, we assume that $\phi_{\alpha\alpha} = \mathrm{id}$ for all α, and we assume that for all α, β, γ we have that

$$\phi_{\alpha\gamma}(x) \circ \phi_{\gamma\beta}(x) \circ \phi_{\beta\alpha}(x) = \mathrm{id},$$

whenever $x \in U_{\gamma\beta\alpha} \neq \emptyset$. On the disjoint union of the sets $U_\alpha \times \mathbb{R}^n$ we define a relation $\sim$ by declaring that $(x, v)_\alpha \sim (y, w)_\beta$ if and only if $x = y$ and $\phi_{\beta\alpha}(x)(v) = w$.

(a) Show that $\sim$ is an equivalence relation. (Hint: show first that $\phi_{\alpha\beta} \circ \phi_{\beta\alpha} = \mathrm{id}$ for all α, β.)

(b) Equip

$$E = (\sqcup_\alpha (U_\alpha \times \mathbb{R}^n)) / \sim$$

with the quotient topology, and define a natural projection $p : E \to M$. Show that for each $x \in M$ we have that $E_x = p^{-1}(x)$ has a natural structure of an n-dimensional real vector space.

Solution: We define a map $p : E \to M$ by $p([(x, v)_\alpha]) = x$. This is clearly well defined. For a fixed α with $x \in U_\alpha$, the map $l_x^\alpha : \mathbb{R}^n \to E_x$ defined by $l_x^\alpha(v) = [(x, v)_\alpha]$ is a bijection; surjective because for any $(x, w)_\beta$ we have that $(x, w)_\beta \sim (x, \phi_{\alpha\beta}(x)(w))_\alpha$, and injective because $\phi_{\alpha\alpha} = \mathrm{id}$. Give E_x a linear structure by declaring that l_x^α is an isomorphism, and check that this is independent of α since the $\phi_{\alpha\beta}(x)$'s are linear maps.

(c) Show that E has the natural structure of a smooth manifold. (Hint: for each α, define a local chart by $[(x, v)_\alpha] \mapsto (\phi_\alpha(x), v)$.)

Solution:

A few words about the topology on E. We claim first that

$$\pi : \sqcup_\alpha (U_\alpha \times \mathbb{R}^n) \to E$$

is an open map. For let $W \subset \sqcup_\alpha(U_\alpha \times \mathbb{R}^n)$ be an open set; this means that $W_\alpha = W \cap (U_\alpha \times \mathbb{R}^n)$ is an open set for each α. For any β with $U_{\alpha\beta} \neq \emptyset$ we have that

$$\pi^{-1}(\pi(W_\alpha)) \cap (U_\beta \times \mathbb{R}^n) = \{(x,w)_\beta; (x,v)_\alpha \in W_\alpha, x \in U_{\alpha\beta}, w = \phi_{\beta\alpha}(x)(v)\},$$

and this is open in $U_\beta \times \mathbb{R}^n$. Taking the union over all β, and then repeating for all α gives the claim. Note also that $p : E \to M$ is continuous.

Note further that since $\phi_{\alpha\alpha} = \mathrm{id}$ we have that $\pi : U_\alpha \times \mathbb{R}^n \to E$ is an injection, hence by the openness a homeomorphism onto its image for each α. Defining $\Phi_\alpha : U_\alpha \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ by $\Phi_\alpha((x,v)_\alpha) = (\phi(x), v)$ we see that $U_\alpha \times \mathbb{R}^n$ is Euclidean, hence E is locally Euclidean.

Let $[(x,v)_\alpha], [(y,w)_\beta] \in E$ be distinct. If $x \neq y$ there are disjoint open sets separating them since M is Hausdorff and p is continuous. If $x = y$ there are disjoint open sets separating them since $U_\alpha \times \mathbb{R}^n$ is Hausdorff.

Since M is second countable we may assume that $\{U_\alpha\}$ is countable, and since $U_\alpha \times \mathbb{R}^n$ is second countable it follows that E is second countable.

This shows that E is a topological manifold, and since the transition maps

$$\Phi_\beta \circ \Phi_\alpha^{-1} : \phi_\alpha(U_{\alpha\beta}) \times \mathbb{R}^n \to \phi_\beta(U_{\alpha\beta}) \times \mathbb{R}^n$$

are given by

$$\Phi_\beta \circ \Phi_\alpha^{-1}(r,v) = (\phi_\beta \circ \phi_\alpha^{-1}(r), \phi_{\beta\alpha}(\phi_\alpha^{-1}(r))(v))$$

which are all smooth, we see that E has the structure of a smooth manifold.

(d) Such an object E is called a *vector bundle* over M. Show that TM is a vector bundle.

Solution: Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas for M. For $\phi_\alpha = (x_1, ..., x_n)$ and $\phi_\beta = (y_1, ..., y_n)$ we set $\phi_{\beta\alpha}(x) = (\frac{\partial y_i}{\partial x_j})(x)$. Check that the compatibility conditions above are satisfied. By the formulas for transition maps on the tangent bundle, the vector bundle E constructed above becomes isomorphic and diffeomorphic to the tangent bundle.