Projective spaces

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1 Real projective spaces

Let $\mathbb{R}^\times$ be the multiplicative group of non-zero real numbers. For $n \geq 0$ we introduce an equivalence relation in $\mathbb{R}^{n+1} - \{0\}$ by declaring that $x \sim y$ if and only if $x = ty$ for some $t \in \mathbb{R}^\times$. The quotient space $\mathbb{RP}^n$ is called **real projective n–space**. The equivalence class of a point $(x_0, \ldots, x_n)$ in $\mathbb{R}^{n+1} - \{0\}$ will be denoted by $[x_0, \ldots, x_n]$. Note that there is a bijective correspondence between $\mathbb{RP}^n$ and the set of lines in $\mathbb{R}^{n+1}$ through the origin.

We are going to show that $\mathbb{RP}^n$ is a compact connected n–manifold.

Lemma 1.1 *The projection*

$$\pi : \mathbb{R}^{n+1} - \{0\} \to \mathbb{RP}^n$$

is an open map.

Proof. Let U be an open subset of $\mathbb{R}^{n+1} - \{0\}$. For every $t \in \mathbb{R}^\times$ the set

$$tU := \{tx \,|\, x \in U\}$$

is an open subset of $\mathbb{R}^{n+1} - \{0\}$. Therefore,

$$\pi^{-1}\pi(U) = \bigcup_{t \in \mathbb{R}^\times} tU$$

is open. Hence, $\pi(U)$ is open in $\mathbb{RP}^n$ by definition of the quotient topology. □

Lemma 1.2 *Let $f : X \to Y$ be a surjective, open, and continuous map between topological spaces. Let $\mathcal{B}$ be a basis for the topology on X. Then*

$$\{f(U) \,|\, U \in \mathcal{B}\}$$

is a basis for the topology on Y.

Proof. Let $V \subset Y$ be open and $y \in V$. Choose $x \in f^{-1}(y)$. Since $f^{-1}(V)$ is open, there is a basis element $U \in \mathcal{B}$ such that

$$x \in U \subset f^{-1}(V).$$

It follows that

$$y \in f(U) \subset V. \qquad \square$$

Lemma 1.3 $\mathbb{RP}^n$ *is second countable.*

Proof. This follows from Lemmas 1.1 and 1.2 since $\mathbb{R}^{n+1} - \{0\}$ is second countable. $\square$

Lemma 1.4 $\mathbb{RP}^n$ *is Hausdorff.*

Proof. Let x, y be points in $\mathbb{R}^{n+1} - \{0\}$ representing distinct points in $\mathbb{RP}^n$. Then x, y are linearly independent, so there are linear maps $S, T : \mathbb{R}^{n+1} \to \mathbb{R}$ such that

$$Sx = 1 = Ty, \quad Sy = 0 = Tx.$$

Let

$$A := \{z \in \mathbb{R}^{n+1} - \{0\} \mid |Sz| > |Tz|\},$$
$$B := \{z \in \mathbb{R}^{n+1} - \{0\} \mid |Sz| < |Tz|\}.$$

Then A, B are disjoint neighbourhoods of x, y. Since A and B are both saturated, their images $\pi(A)$ and $\pi(B)$ are disjoint neighbourhoods of $\pi(x)$ and $\pi(y)$. $\square$

Theorem 1.1 $\mathbb{RP}^n$ *is an n–manifold.*

Proof. For $j = 0, \dots, n$ let

$$U_j := \{[x_0, \dots, x_n] \in \mathbb{RP}^n \mid x_j \neq 0\}.$$

Then $\{U_0, \dots, U_n\}$ is an open covering of $\mathbb{RP}^n$. Let $\phi_j : U_j \to \mathbb{R}^n$ be defined by

$$\phi_j([x_0, \dots, x_n]) := \left(\frac{x_0}{x_j}, \dots, \widehat{\frac{x_j}{x_j}}, \dots, \frac{x_n}{x_j} \right),$$

where the $\widehat{\ }$ indicates a term that is to be omitted. Using Lemma 1.1 it is easy to see that ϕ_j is continuous. In fact, ϕ_j is a homeomorphism, the inverse map $\mathbb{R}^n \to U_j$ being given by

$$\phi_j^{-1}(y_1, \dots, y_n) = [y_1, \dots, y_j, 1, y_{j+1}, \dots, y_n].$$

This shows that every point in $\mathbb{RP}^n$ has a neighbourhood homeomorphic to $\mathbb{R}^n$. Since we have already verified that $\mathbb{RP}^n$ is second countable and Hausdorff, it follows that $\mathbb{RP}^n$ is an n–manifold. □

Theorem 1.2 $\mathbb{RP}^n$ *is compact and connected.*

Proof. For $n = 0$ the statement is trivial since $\mathbb{RP}^0$ consists of just one point. For $n \geq 1$ observe that the projection map

$$S^n \to \mathbb{RP}^n$$

is surjective and continuous. Since S^n is compact and connected, it follows that $\mathbb{RP}^n$ has the same properties. □

Theorem 1.3 $\mathbb{RP}^n$ *is homeomorphic to the quotient space* $S^n/\pm 1$ *obtained by identifying antipodal points in* S^n*.*

Proof. Let $\rho : S^n \to \mathbb{RP}^n$ be the projection. Two point x, y in S^n have the same image under ρ precisely when $x = \pm y$. Therefore, ρ induces a bijective and continuous map

$$f : S^n/\pm 1 \to \mathbb{RP}^n.$$

Because $S^n/\pm 1$ is compact and $\mathbb{RP}^n$ is Hausdorff, the map f is a homeomorphism. □

Example $\mathbb{RP}^1$ is homeomorphic to S^1. To see this, we regard S^1 as the unit circle in the complex plane and consider the map

$$f : S^1 \to S^1, \quad z \mapsto z^2.$$

Arguing as in the proof of Theorem 1.3 we find that f descends to a homeomorphism $S^1/\pm 1 \to S^1$.

Theorem 1.4 *For* $n \geq 1$ *the map*

$$\mathbb{RP}^{n-1} \to \mathbb{RP}^n, \quad [x_0, \ldots, x_{n-1}] \mapsto [x_0, \ldots, x_{n-1}, 0]$$

is an embedding.

Proof. It is convenient to identify $\mathbb{R}^n$ with $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$. The inclusion map $\mathbb{R}^n - \{0\} \to \mathbb{R}^{n+1} - \{0\}$ induces an injective continuous map

$$g : \mathbb{RP}^{n-1} \to \mathbb{RP}^n.$$

Because $\mathbb{R}^n - \{0\}$ is a closed subset of $\mathbb{R}^{n+1} - \{0\}$, the map g is closed, hence an embedding. □

Henceforth, we identify $\mathbb{RP}^{n-1}$ with its image in $\mathbb{RP}^n$.

Theorem 1.5 *The complement* $\mathbb{RP}^n - \mathbb{RP}^{n-1}$ *is homeomorphic to* $\mathbb{R}^n$.

Proof. The complement is just the domain of the homeomorphism $\phi_n : U_n \to \mathbb{R}^n$ constructed in the proof of Theorem 1.1. □

2 Complex projective spaces

The *complex projective space* $\mathbb{CP}^n$ is defined in the same way as $\mathbb{RP}^n$, except that the real numbers $\mathbb{R}$ are replaced with the complex numbers $\mathbb{C}$. To be explicit, $\mathbb{CP}^n$ is the quotient of $\mathbb{C}^{n+1} - \{0\}$ obtained by identifying two points w, z whenever $w = tz$ for some non-zero complex number t. Arguing as before we find that $\mathbb{CP}^n$ is a $(2n)$–manifold. The reason why the dimension is $2n$ is that the homeomorphisms ϕ_j now take values in $\mathbb{C}^n$, which we identify with $\mathbb{R}^{2n}$ as a real vector space via the map

$$(x_1 + iy_1, \dots, x_n + iy_n) \mapsto (x_1, y_1, \dots, x_n, y_n).$$

The projection

$$S^{2n+1} \to \mathbb{CP}^n$$

is continuous and surjective, hence $\mathbb{CP}^n$ is compact and connected. Taking $n = 1$ we get the *Hopf map*

$$S^3 \to \mathbb{CP}^1.$$

The space $\mathbb{CP}^1$ is known as the *Riemann sphere* because of the following result.

Theorem 2.1 $\mathbb{CP}^1$ *is homeomorphic to* S^2.

Proof. As in the proof of Theorem 1.1 let

$$U_1 := \{[z_0, z_1] \in \mathbb{CP}^1 \mid z_1 \neq 0\}.$$

The complement of U_1 consists only of the point $[1, 0]$, and we have a homeomorphism

$$\phi_1 : U_1 \to \mathbb{C}, \quad [z_0, z_1] \mapsto z_0/z_1.$$

On the other hand, stereographic projection provides a homeomorphism

$$\tau : S^2 - \{N\} \to \mathbb{R}^2,$$

where N is the "north pole". Since $\mathbb{CP}^1$ and S^2 are compact Hausdorff spaces, the composite homeomorphism

$$\tau^{-1} \circ \phi_1 : U_1 \to S^2 - \{N\}$$

extends to a homeomorphism $\mathbb{CP}^1 \to S^2$ (see [1, Theorem 29.1]). □

References

[1] J. R. Munkres. *Topology*. Prentice Hall, 2nd edition, 2000.