8^{th} May, 2024

MAT4520 - Spring 2024

Mandatory assignment 1 of 1

Submission deadline

Thursday 18^{th} April 2024, 14:30 in Canvas (canvas.uio.no).

Instructions

Note that you have **one attempt** to pass the assignment. This means that there are no second attempts.

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) no later than the same day as the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

Problem 1.

In this problem we are going to construct a smooth manifold $G(k,n)$ - a so called Grassmannian - whose elements consists of the set of k-dimensional linear subspaces of $\mathbb{R}^n$. This is a generalization of a construction we have already considered, namely the construction of $\mathbb{RP}^{n-1} = G(1,n)$.

A k-plane L in $\mathbb{R}^n$ may be determined by choosing a basis $a_1, ..., a_k$ for L, and so L may be represented by an $n \times k$-matrix

$$A = [a_1 a_2 \cdots a_k]$$

which we identify with an element of $\mathbb{R}^{n \cdot k}$. Since the vectors a_j are linearly independent we have that this matrix has rank k[1]. Letting $F(k,n)$ denote the set of $k \times n$-matrices of rank k we have that any $B \in F(k,n)$ determines a k-plane. However, there is a large set of elements $B \in F(k,n)$ that determines the same k-plane. We introduce an equivalence relation $\sim$ on $F(k,n)$ by declaring $A \sim B$ if and only if A and B determine the same k-plane. The manifold $G(k,n)$ will be constructed as the quotient $F(k,n)/\sim$ (where $F(k,n)$ has the topology induced by the identification with $\mathbb{R}^{n \cdot k}$).

(i) Explain why $A \sim B$ if and only if there exists $g \in GL(k, \mathbb{R})$ such that $B = Ag$.

(ii) Explain why for each fixed $g \in GL(k, \mathbb{R})$ the map $A \mapsto Ag$ is a homeomorphism on $F(k,n)$. Prove that $\sim$ is an *open* equivalence relation.

(iii) Prove that $G(k,n)$ is *second countable*.

(iv) Prove that $G(k,n)$ is *Hausdorff*.

Solution (i)

Suppose that $a = \{a_1, ..., a_k\}$ and $b = \{b_1, ..., b_k\}$ span the same k-plane L. Then there exist real numbers g_{ij} such that we have that

$$b_j = \sum_i g_{ij} a_i \tag{1}$$

for $j = 1, ..., k$. If we let $G = [g_{ij}]$ we then have that $B = Ag$. If the rank of g were less than k, then the rank of Ag would be less than k, but this is not the case. So the rank of g is k, and so g is invertible.

[1] If for a multi-index $I = (i_1, ..., i_k), i_1 < i_2 < \cdots < i_k$, we let A_I denote the $k \times k$-matrix whose rows are the I-rows of A, we have that the rank of A is k if and only if $det A_I \neq 0$ for at least one I.

On the other hand, if such a matrix g exists we would have (1). So $\mathrm{Span}(b) \subset \mathrm{Span}(a)$. But we would also have $A = Bg^{-1}$ and so $\mathrm{Span}(a) \subset \mathrm{Span}(b)$.

Solution (ii)

The map $A \mapsto Ag$ is clearly a continuous map. It has a continuous inverse given by the map $B \mapsto Bg^{-1}$.

Now if $\pi : F(k,n) \to G(k,n)$ denotes the projection map, and if $U \subset F(k,n)$ is open, we have that

$$\pi^{-1}(\pi(U)) = \bigcup_{g \in GL(k,\mathbb{R})} Ug,$$

which is a union of open sets, since each Ug is an open set. By the definition of the quotient topology we have that $\pi(U)$ is open.

Solution (iii)

Since $F(k,n)$ is second countable it follows from Corollary 7.10 in the course book that the quotient $G(k,n)$ is second countable.

Solution (iv)

According to Theorem 7.7 in the course book it suffices to show that the graph of $\sim$ is closed in $F(k,n) \times F(k,n)$. Representing elements in this product by a matrix $[AB]$ the graph is given as the solution set of all $(k+1) \times (k+1)$-minors having determinant zero. This is a closed set.

We would now like to find a smooth atlas for $G(k,n)$. For each multi-index I we let

$$V_I = \{A \in F(k,n) : \det(A_I) \neq 0\}$$

and note that $\{V_I\}$ is an open cover of $F(k,n)$. Since $\sim$ is open we have that each $U_I = \pi(V_I)$ is an open subset of $G(k,n)$ and we have that $\{U_I\}$ is an open cover of $G(k,n)$.

Next we initially consider the fixed multi-index $I = (1, 2, ..., k)$. For each $A \in V_I$ we have that

$$A \sim AA_I^{-1} = \begin{bmatrix} \mathrm{Id} \\ A_{\widehat{I}}A_I^{-1} \end{bmatrix}$$

where $\widehat{I} = (k+1, ..., n)$. Therefore any equivalence class may be represented by an element of $\mathbb{R}^{(n-k)\times k}$.

(v) Show that the map

$$\tilde{\phi}_I : V_I \to \mathbb{R}^{(n-k)\times k}$$

induces a homeomorphism

$$\phi_I : U_I \to \mathbb{R}^{(n-k)\times k}$$

(vi) Define similarly maps ϕ_I for all multi-indices I. Show that the transition maps $\phi_J \circ \phi_I^{-1}$ are smooth for multi-indices I, J.

(vii) Conclude that we have given $G(k, n)$ the structure of a smooth manifold.

Solution (v)

By Proposition 7.1 in the course book we have that the induced map ϕ_I is continuous. The map

$$B \overset{\psi}{\mapsto} \begin{pmatrix} \mathrm{Id} \\ B \end{pmatrix}$$

is a continuous map from $\mathbb{R}^{(n-k)\times k}$ to $F(k, n)$ and so the map $\pi \circ \psi$ is continuous, and it is the inverse of ϕ_I.

Solution (vi)

For a multiindex I and $A \in V_I$ we let $\tilde{\phi}_I(A)$ be the $(n-k) \times k$-matrix obtained by deleting the I-rows of AA_I^{-1}. Note that $(AA_I^{-1})_I = \mathrm{Id}$. Similar to above, the map induces a homeomorphism $\phi_I : U_I \to \mathbb{R}^{(n-k)\times k}$, whose inverse is induced by the map $\psi_I : \mathbb{R}^{(n-k)\times k} \to V_I$ defined by inserting the identity matrix along the I-rows. Now if $A \in V_I \cap V_J$ we have that $(\psi_I \circ \phi_I)(A) \in V_J$. So we have that $\psi(\phi_I(U_I \cap U_J)) \in V_J$. We have that

$$\phi_J \circ \phi_I^{-1}(B) = \tilde{\phi}_J(\psi_I(B)(\psi_I(B)_J)^{-1})$$

and this is a smooth map.

Solution (vii)

We have shown that $G(k, n)$ has been given the structure of a locally Euclidean, Hausdorff, second countable topological space, and that $G(k, n)$ has a C^∞-compatible atlas.