MAT4520

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1. Some Exercises

5.4 We will prove the following. Suppose that (U, ϕ) is a chart on a manifold M. Then for any open set $V \subset U$ we have that (V, ϕ_V) is a coordinate chart.

First of all, since $V \subset U$ is open, we have that $\phi(V)$ is open and

$$\phi_V : V \to \phi(V)$$

is a homeomorphism. Suppose then that (W, ψ) is another chart. Then

$$\psi \circ \phi^{-1} : \phi(U) \cap W \to \psi(U \cap W)$$

is smooth. Then

$$\psi \circ \phi_V^{-1} : \phi_V(V) \cap W \to \psi(U \cap W)$$

is smooth. Similarly $\phi_V \circ \psi^{-1}$ becomes smooth.

6.1

a) The atlases are not compatible since $\psi \circ \phi^{-1}(x) = x^{1/3}$ which is not smooth.

b) Consider the map $f : \mathbb{R} \to \mathbb{R}'$ given by $f(x) = x^3$. Using the two charts we have that

$$\psi \circ f \circ \phi^{-1}(x) = x$$

which is smooth. Similarly one shows that f^{-1} is smooth.

6.4

We need to check where the Jacobian matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 2x & 2y & 2z \\ 0 & 0 & 1 \end{pmatrix}$$

has maximal rank. We see that the map serves a a local coordinate system except at the points $y = 0$.

7.2

Follow the hint.

7.6

Show first that the induced equivalence relation is open, and that the topology is Hausdorff. Then the orbit space S is Hausdorff and second countable. Consider the open sets $U_1 = (0, 2\pi)$ and $U_2 = (\pi, 3\pi)$ in $\mathbb{R}$. Then the quotient maps $\pi_j : U_j \to V_j = \pi_j(U_j) \subset S$ are homeomorphisms. The pair $\{V_1, V_2\}$ covers S and so S is locally euclidean. Now we have that $V_1 \cap V_2$ consists of two connected components, and the transition map $\pi_2 \circ \pi_1^{-1}$ is given by the identity map and the map $x \mapsto x + 2\pi$.

8.1

In the book.

8.2

We have seen that for a general smooth map $F : \mathbb{R}^n \to \mathbb{R}^m$ the differential $F_{*,p}$ is represented by the Jaconian matrix $JF(p)$. For a linear map $F = L$ this Jacobian coincides with L itself.

9.1

The Jacobian of f is

$$\begin{pmatrix} 3x^2 - 6y & 2y - 6x \end{pmatrix}$$

If the rank is 0 we have that $y = 3x$ and $y = (1/2)x^2$. So we would have $0 = 3x - (1/2)x^2 = x(3 - (1/2)x)$. So the rank is 1 outside the two points $(0, 0)$ and $(6, 18)$.

9.2

The solution set is the 0-level set of the function

$$f(x, y, z, w) = x^5 + y^5 + z^5 + w^5 - 1$$

The Jacobian of this map is

$$5\begin{pmatrix} x^4 & y^4 & z^4 & w^4 \end{pmatrix}$$

The rank of this matrix is zero only if $x = y = z = w = 0$, but in that case we are not on the 0-level set of f. So the level set if a regular submanifold.

9.3

We consider the map $f : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f(x, y, z) = (x^3 + y^3 + z^3 - 1, z - xy)$$

such that the solution set is the 0-level set of f. The Jacobian of the map f is given by

$$\begin{pmatrix} 3x^2 & 3y^2 & 3z^2 \\ -y & -x & 1 \end{pmatrix}$$

If the rank of this matrix less than 2 we need $y^3 - x^3 = 0$ which means $x = y$. We would further have $y^2 = -z^2 y$ and so $y = -z^2$. We would have that $z - z^4 = z(1 - z^3) = 0$. If $z = 1$ then we would need $x = y = 0$, but then $z - xy \neq 0$. If $z = 0$ we have that $2x^3 = 1$ and so $x = y = 2^{-1/3}$ but then $z - xy \neq 0$. So the rank of f is two along the 0-level set, and so the level set is a regular submanifold.

11.1

For a point $p \in S^n$ the tangent space T_pS^n is an n-dimensional subspace of $T_p\mathbb{R}^{n+1}$. Any $\nu \in T_pS^n$ is induced by a smooth curve $c : (-\epsilon, \epsilon) \to S^n$. Setting $f(x) = \sum_{i=1}^{n+1}(x^i)^2$ we have that $f(c(t)) \equiv 1$, and so $\frac{d}{dt}f(c(t)) \equiv 0$. By the chain rule we have that

$$\frac{d}{dt}f(c(t)) = \nabla f(c(t)) \cdot \dot{c}(t)$$

and so

$$\frac{d}{dt}f(c(t))|_{t=0} = \nabla f(p) \cdot \dot{c}(0) = 2\sum_{i=1}^{n+1} \cdot p^j \dot{c}_j(0) = 0$$

This shows that T_pS^n is a subspace of the vector space of tangent vectors

$$X_p = \sum a^i \frac{\partial}{\partial x^i}|_p$$

such that $\nabla f(p) \cdot a = 0$. The tangent space T_pS^n has dimension n, so it has to be all of this space.

12.1

Let $\nu_p, \nu_q \in TM$ be distinct. If $p \neq q$ there are disjoint open sets U, V with $p \in U, q \in V$. Then TU and TV are disjoint open sets containing $\nu_p, \nu_q \in TM$ respectively.

If $p = q$ choose a coordinate chart (U, ϕ) with $p \in U$. Then $\nu_1 = \tilde{\phi}(\nu_p)$ and $\nu_2 = \tilde{\phi}(\nu_q)$ are distinct, so there are disjoint open subsets V, W in $\phi(U) \times \mathbb{R}^n$ of ν_1 and ν_2 resepectively. So $\tilde{\phi}^{-1}(V)$ and $\tilde{\phi}^{-1}(W)$ are disjoint open subsets of TM separating the original two points.

12.2

(a) We have seen that the map $\tilde{\psi} \circ \tilde{\phi}$ is the map

$$\tilde{\psi} \circ \tilde{\phi}(x, a) = ((\psi \circ \phi^{-1})(x), J(\psi \circ \phi^{-1})(x)a) = ((\psi \circ \phi^{-1})(x), [\frac{\partial y^i}{\partial x^j}](x)a)$$

The n last components, indexed by i, become

$$\sum_j \frac{\partial y^i}{\partial x^j}(x) \cdot a_j$$

The Jacobian consists of four $n \times n$-blocks. The upper left block is the map $\psi \circ \phi^{-1}$ differentiated with respect to the x-varibles, which is $[\frac{\partial y^i}{\partial x^j}](x)$. The map $\psi \circ \phi^{-1}$ does not depend on a so the upper right block is zero. The lower left block you get from differentiating the n last components with respect to the x^js so this is the matrix

$$[\sum_j \frac{\partial^2 y^i}{\partial x^j \partial x^k}(x) \cdot a_j]_{1\leq i,k\leq n}$$

The lower right block you get by differentiating the last n components with respect to the a^is, and you get back $[\frac{\partial y^i}{\partial x^j}](x)$.

(b) The Jacobian determinant is the determinant of the Jabian described in (a) wich is the product of the determinants of the upper left and the lower right determinants.

18.3

On the one hand we have that

$$\begin{aligned} F^*\omega \wedge F^*\tau(X_1,...,X_{k+l}) &= \frac{1}{k!l!}\sum_\sigma \mathrm{sgn}\sigma \cdot F^*\omega \otimes F^*\tau(X_{\sigma(1)},...,X_{\sigma(\tau)}) \\ &= \frac{1}{k!l!}\sum_\sigma \mathrm{sgn}\sigma \cdot \omega \otimes \tau(F_*X_{\sigma(1)},...,F_*X_{\sigma(\tau)}) \end{aligned}$$

One the other hand we have that

$$\begin{aligned} F^*(\omega \wedge \tau)(X_1,...,X_{k+l}) &= \omega \wedge \tau(F_*X_1,...,F_*X_{k+l}) \\ &= \frac{1}{k!l!}\sum_\sigma \mathrm{sgn}\sigma \cdot \omega \otimes \tau(F_*X_{\sigma(1)},...,F_*X_{\sigma(\tau)}) \end{aligned}$$

19.2

We are given

$$F(x,y) = (x^2+y^2, xy) \text{ and } \omega = udu + vdv$$

We have that

$$\begin{aligned} F^*\omega &= F^*uF^*du + F^*vF^*dv \\ &= u \circ FdF^*u + v \circ FdF^*v \\ &= (x^2+y^2)d(x^2+y^2) + xyd(xy) \\ &= (x^2+y^2)(2xdx+2ydy) + xy(ydx+xdy) \\ &= (2x^3+xy^2)dx + (2y^3+x^2y)dy \end{aligned}$$

19.3

We are given

$$\tau = \frac{-ydx + xdy}{x^2+y^2} \text{ and } \gamma(t) = (\cos(t), \sin(t)), t \in \mathbb{R}$$

We have that

$$\begin{aligned} \gamma^*\tau &= (-\sin(t)d(\cos(t)) + \cos(t)d(\sin(t)))dt \\ &= (\sin^2(t) + \cos^2(t))dt = dt \end{aligned}$$

21.2

Suppose that $[(X_1, ..., X_n)]$ is continuous. Then for each point $q \in M$ there is a coordinate neighborhood $(U, \phi = (x^1, ..., x^n))$ such that

$$[(X_1, ..., X_n)] \sim [(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n})]$$

on U. But then

$$[(\phi_* X_1, ..., \phi_* X_n)] \sim [(\phi_* \frac{\partial}{\partial x^1}, ..., \phi_* \frac{\partial}{\partial x^n})] = [(\frac{\partial}{\partial r^1}, ..., \frac{\partial}{\partial r^n})]$$

On the other hand, if

$$[(\phi_* X_1, ..., \phi_* X_n)] \sim [(\frac{\partial}{\partial r^1}, ..., \frac{\partial}{\partial r^n})]$$

then

$$[(X_1, ..., X_n)] \sim [(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n})]$$

by applying $(\phi^{-1})_*$

22.4 This is a $*$-problem.

23.1 Letting E denote the domain enclosed by the ellipse. There is an orientation preserving diffeomorphism $F : B^2 \to E$ defined by

$$F(x, y) = (ax, by)$$

We have that

$$A(E) = \int_E du \wedge dv = \int_{B^2} F^*(du \wedge dv) = \int_{B^2} abdx \wedge dy = ab\pi$$

23.3 This is a $*$-problem.

24.1 Let $\omega = df$ be a smooth 1-form an a compact manifold M. Since M is compact we have that f has a global maximum at a point p. Choosing a coordinate chart (U, ϕ) near p we have that $\frac{\partial f}{\partial x^j}(p) = 0$ for $j = 1, ..., n$. We have that

$$\omega(p) = \sum_j \frac{\partial f}{\partial x^j}(p) \frac{\partial}{\partial x^j}|_p = 0$$

25.1 Prove Proposition 25.2. (misprint)

25.2 Prove Proposition 25.3. (misprint)

(ii) Consider an exact sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

It is standard that g induces an injective linear map $\tilde{g} : B/\mathrm{Im} f \to C$ since $\mathrm{Im} f = \mathrm{Ker}(g)$. Since g is surjective we have that $\tilde{g}$ is surjective, and so $\tilde{g}$ is an isomorphism.

25.3 Prove the exactness of the cohomology sequence (25.4) at $H^k(A)$ and $H^k(B)$.

26.1

Prove exactness of the sequence

$$0 \to \Omega^k(M) \xrightarrow{i} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{j} \Omega^k(U \cap V) \to 0$$

at $\Omega^k(U) \oplus \Omega^k(V)$.

$\mathrm{Im} i \subset \mathrm{Ker} j$: For $\omega \in \Omega^k(M)$ we have that $i(\omega) = (\omega|_U, \omega|_V)$. By definition

$$j(i\omega) = \omega|_{U \cap V} - \omega|_{U \cap V} = 0.$$

$\mathrm{Ker} j \subset \mathrm{Im} i$: Suppose that $(\omega, \tau) \in \mathrm{Ker} j$. Then $\omega|_{U \cap V} = \tau|_{U \cap V}$, and so there is a $\sigma \in \Omega^k(M)$ such that $\sigma|_U = \omega, \sigma|_V = \tau$.

27.1

There is a homotopy equivalence $f : M \to N$ with homotopy inverse $g : N \to M$, and a homotopy equivalence $h : N \to P$ with homotopy inverse $k : P \to N$. There is a homotopy $F : M \times \mathbb{R} \to M$ such that $F(\cdot, 0) = g \circ f, F(\cdot, 1) = \mathrm{Id}$, and a homotopy $G : N \times \mathbb{R} \to N$ such that $G(\cdot, 0) = k \circ h, F(\cdot, 1) = \mathrm{Id}$.

We now want to show that $g \circ k \circ h \circ f : M \to M$ is homotopic to the identity map.

Step 1. Consider the homotopy $g \circ G_t \circ f$. We have that

$$g \circ G_0 \circ f = g \circ k \circ h \circ f \text{ and } g \circ G_1 \circ f = g \circ f$$

So

$$g \circ k \circ h \circ f \sim g \circ f$$

Step 2. By assumption we have $g \circ f \sim \text{Id}$. Since $\sim$ is an equivalence relation (proved in class) we have that $g \circ k \circ h \circ f \sim \text{Id}$.

In a similar fashion we may show that $h \circ f \circ g \circ k \sim \text{Id}$.

27.2 Let $p, q \in M$ where M is contractible. By definition, there is a smooth map $F : M \times \mathbb{R} \to \mathbb{R}$ such that $F(\cdot, 0) = \text{Id}$ and $F(\cdot, 1) = \{x\}$ for some point $x \in M$.

Then $\gamma(t) = F(p, t), t \in [0, 1]$, is a continuous path such that $\gamma(0) = p, \gamma(1) = x$, and $\tilde{\gamma}(t) = F(q, t), t \in [0, 1]$, is a continuous path such that $\tilde{\gamma}(0) = q, \gamma(1) = x$. We define a continuous path $\kappa : [0, 2] \to M$ by setting $\kappa(t) = \gamma(t)$ for $t \in [0, 1]$ and $\kappa(t) = \tilde{\gamma}(1 - t/2)$ for $t \in [1, 2]$. Then κ is a continuous path connecting p and q.

27.3 In this case it is easy to write down a retraction:

$$F(p, v, t) = (p, (1 - t)v)$$

We may also then show that $S^1 \subset \mathbb{R}^2 \setminus \{0\}$ is a deformation retract. There is a diffeomorphism $G : \mathbb{R}^2 \setminus \{0\} \to S^1 \times \mathbb{R}$ defined by

$$G(x) = (x/\|x\|, \log \|x\|)$$

sending S^1 to $S^1 \times \{0\}$. Its inverse is given by $H(p, v) = e^v p$. We have that

$$H \circ F_t \circ G(x) = H(x/\|x\|, (1 - t) \log \|x\|) = e^{(1-t) \log \|x\|} x/\|x\|$$

References

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