These notes on group theory contains background material used in the book "Algebraic Topology" by Allen Hatcher. In particular all these notions are used repeatedly in section 1.2.

Abelianization of Groups

Definition 1 For any group G define the **commutator subgroup** $[G,G]$ as the normal subgroup generated by the commutators:

$$[a,b] = aba^{-1}b^{-1}, \qquad a,b \in G.$$

Define the quotient group

$$G^{\mathrm{ab}} = G/[G,G].$$

The group G^{ab} is called the **Abelianization** of G. $\Diamond$

Note that the normal subgroup generated by a set of elements is defined to be the smallest normal subgroup containing them.

Remark 2 Since any group homomorphism $f\colon G \to H$ sends the commutator subgroup to the commutator subgroup ($f([a,b]) = [f(a), f(b)]$) we get an induced group homomorphism on the quotients:

$$f^{\mathrm{ab}}\colon G^{\mathrm{ab}} \to H^{\mathrm{ab}}.$$

These behave well under composition: $(f_1 \circ f_2)^{\mathrm{ab}} = f_1^{\mathrm{ab}} \circ f_2^{\mathrm{ab}}$. $\Diamond$

Lemma 3 The group G^{ab} is Abelian. If N is another normal subgroup and the quotient G/N is Abelian then $[G,G] \subset N$. Furthermore, isomorphic groups have isomorphic Abelianizations. $\Diamond$

Proof:
- In $G/[G,G]$ we have $[b][a] = [ba] = [(aba^{-1}b^{-1})ba] = [ab] = [a][b]$.
- In G/N: if $[a][b] = [b][a], \forall a,b \in G$ then $[aba^{-1}b^{-1}] \in N, \forall a,b \in G$.

Let $f\colon G \to G'$ be an isomorphism then $\mathrm{Id}_{G^{\mathrm{ab}}} = (f \circ f^{-1})^{\mathrm{ab}} = f^{\mathrm{ab}} \circ (f^{-1})^{\mathrm{ab}}$, and vice versa for the identity on $(G')^{\mathrm{ab}}$. So f^{ab} is an isomorphism. $\square$

Example 4 The Abelianization of $F_n = *_{i=1}^{n}\mathbb{Z}$ is isomorphic to $\mathbb{Z}^n = \times_{i=1}^{n}\mathbb{Z}$.

In F_n we denote the generators $g_1, \ldots, g_n$. Define

$$e_i\colon F_n \to \mathbb{Z}$$

as the group homomorphism that sums all the exponents of g_i in a word in F_n. This is well-defined because reducing a word does not change this sum, and it is a homomorphism because juxtaposition adds the two sums. Use this to define the group homomorphism

$$E = \times_{i=1}^{n} e_i\colon F_n \to \mathbb{Z}^n.$$

Note that this can also be defined as the induced group homomorphism $E = *_{i=1}^{n}(j_i)$, where j_i denotes the inclusion of $\mathbb{Z}$ into $\mathbb{Z}^n$ as the i^{th} factor.

The homomorphism E is obviously surjective. So we get a surjective map of quotients:

$$E^{\mathrm{ab}}\colon F_n^{\mathrm{ab}} \to (\mathbb{Z}^n)^{\mathrm{ab}} = \mathbb{Z}^n.$$

This map is also injective. Indeed, the $[g_i]$'s generate F_n^{ab} and they commute; so counting their exponents uniquely determines the element. $\diamondsuit$

The center of a group

Definition 5 For any group G define the **center** $Z(G)$ by

$$Z(G) = \{g \in G \mid gh = hg \text{ for all } h \in G\}.$$

It is easy to check that this is a normal subgroup. $\diamondsuit$

Remark 6 Any group isomorphism $f\colon G \to H$ takes the center $Z(G)$ isomorphically to the center $Z(H)$ and so we have an induced isomorphism on quotients:

$$f'\colon G/Z(G) \to H/Z(H).$$

So we conclude that the isomorphism class of $G/Z(G)$ only depends on the isomorphism class of G. $\diamondsuit$

Lemma 7 If $N \subset Z(G)$ is a (normal) subgroup of G and G/N has trivial center then $N = Z(G)$. $\diamondsuit$

Proof: any element $g \in Z(G) - N$ defines a non-trivial element in $Z(G/N)$ because $[g][h] = [gh] = [hg] = [h][g]$ for all $[h] \in G/N$. $\square$

Presentations of groups

Definition 8 By the free group generated by g_α, $\alpha \in A$ we mean the free product

$$\langle g_\alpha \rangle = *_\alpha G_\alpha,$$

where $g_\alpha \in G_\alpha = \mathbb{Z}$ is the standard generator 1. So words (possibly unreduced) representing elements in $\langle g_\alpha \rangle$ are sequences of integer powers of the g_α's.

For any subset $\{r_\beta\} \subset \langle g_\alpha \rangle$ we define the group

$$\langle g_\alpha \mid r_\beta \rangle = \langle g_\alpha \rangle / N,$$

where N is the normal subgroup generated by the r_β's.

The g_α's are called generators and the r_β's are called relations. $\diamondsuit$

Lemma 9 Define the equivalence relation on the unreduced words in the g_α's by saying that two words are equivalent if they can be joined by a finite sequence of: reductions, inverse reductions, removals of a sub-word r_β and insertions of a sub-word r_β. Then juxtaposition is a well-defined group structure on the set of equivalence classes. Further more this group is isomorphic to $\langle g_\alpha \mid r_\beta \rangle$. $\diamondsuit$

Proof: Since a sequences of the operations on a word x and a sequence on a word y is easily juxtaposed into a sequence of the operations on the juxtaposed word xy it follows that juxtaposition is well-defined on equivalence classes.

It defines a group structure because it is a product preserving quotient of the group $\langle g_\alpha \rangle$.

The kernel of this quotient is a normal subgroup N' containing all the r_β's - so $N \subset N'$, where N is generated by the r_β's. Conversely, the operation of removing or inserting an r_β can be achieved by multiplication with an element from N. Indeed, we see

$$g_1 \cdots g_i g_{i+1} \cdots g_n = (g_1 \cdots g_i r_\beta^{-1} g_i^{-1} \cdots g_1^{-1}) g_1 \cdots g_i r_\beta g_{i+1} \cdots g_n.$$

So we conclude that the kernel is in fact N. $\square$

Remark 10 An isomorphism $G \approx \langle g_\alpha \mid r_\beta \rangle$ is called a **presentation** of G. $\Diamond$

Example 11 $\langle a, b \mid a^n, b^m \rangle \approx \mathbb{Z}_n * \mathbb{Z}_m$. Indeed, the added equivalence from lemma 9 on words in $\langle a, b \rangle$ is just the standard one of removing neutral elements if we think of a and b as generators of $\mathbb{Z}_n$ and $\mathbb{Z}_m$ respectively both instead of $\mathbb{Z}$. $\Diamond$