This is a short note describing some extra details in the proofs of proposition 0.16 and 0.18 in Allen Hatchers book: Algebraic Topology.

Addendum to Proof of Proposition 0.16: Let

$$G_n\colon (X^n \times I) \times I \to X^n \times I$$

be the deformation retraction of $X^n \times I$ to $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I$ described by Hatcher in the proof of proposition 0.16.

To be able to concatenate these, as Hatcher describes, it is convenient to extend them. So define

$$Z_n = X \times \{0\} \cup (X^n \cup A) \times I,$$

for $n \geq -1$, with the convention $X^{-1} = \emptyset$. Notice that

$$Z_n = (X^n \times I) \cup (X \times \{0\}) \cup (A \times I) = (X^n \times I) \cup Z_{-1}.$$

So we extend G_n composed with the inclusion $X^n \times I \to Z_n$ to get

$$G_n'\colon Z_n \times I \to Z_n$$

by putting G_n' equal to the homotopy constantly equal to the identity on the new part. This is continuous by the pasting lemma because G_n is already defined to be this on the intersection $(X^n \times I) \cap Z_{-1} = X^n \times \{0\} \cup A^n \times I$, and both $(X^n \times I)$ and Z_{-1} are closed subsets whose union is Z_n.

We now see that G_n' is a deformation retraction from Z_n to Z_{n-1}, and now we may concatenate these as Hatcher does using G_n' on the sub-interval $[1/2^{n+1}, 1/2^n]$. To get a deformation retraction of

$$X \times I = \bigcup_{n \geq 0} Z_n$$

to

$$X \times \{0\} \cup A \times I = Z_{-1}.$$

The argument that this concatenation is continuous relies on the fact that $X \times I$ is a CW-complex with $n-1$ skeleton inside $X^n \times I$. This is a black box from appendix A, which we will not prove in this course. Notice, however, that this black box is not needed if X is a finite dimensional CW-complex. Indeed, in this case being continuous on $X^n \times I$ implies being continuous on $X \times I$ because $X = X^n$ for some n, and we need only concatenate finitely many deformation retractions. □

Addendum to Proof of Proposition 0.18: This is a proof of the fact that $X_0 \sqcup_f X_1$ is in fact a subspace of $X_0 \sqcup_F (X_1 \times I)$ or equivalently that it is homeomorphic to the subspace $X_0 \sqcup_F (X_1 \times \{0\} \cup A \times I)$.

Since f is F restricted to $A \times \{0\}$ we have an inclusion of sets $X_0 \sqcup_f X_1 \to X_0 \sqcup_F (X_1 \times \{0\} \cup A \times I)$. This is also surjective because everything in $A \times I$ is identified in the quotient with something in X_0. Continuity is easily proved - but the slightly subtle fact is that the inverse is continuous.

This follows if we can prove that a saturated subset $U \subset X_0 \sqcup (X_1 \times \{0\} \cup A \times I)$ is open if and only if its restriction to the subspace $X_0 \sqcup (X_1 \times \{0\})$ is open. Again the direction implying continuity is easy; it follows because it is a subspace.

For the other direction we see that any saturated set C must satisfy: $C = (C \cap (X_1 \sqcup (X_0 \times \{0\}))) \cup F^{-1}(C \cap X_1)$. Take C to be the complement of U, and the assumption now translate to $C \cap (X_1 \sqcup (X_0 \times \{0\}))$ being closed. This implies that $F^{-1}(C \cap X_1)$ is closed in $A \times I$ and hence also closed in $X_1 \sqcup (X_0 \times \{0\} \cup A \times I)$. So C is the union of two closed sets and thus closed - so U is open.

Again we note that we only used the black box from appendix A in the proof of this proposition if X_1 is not finite dimensional. $\square$

Remark 1 In this course we will only use proposition 0.16, 0.17 and 0.18 in the finite dimensional case. $\lozenge$