Problem 7

Let X be a compact, convex set in $\mathbb{R}^n$ with $\overset{o}{X} \neq \varnothing$. Replacing X with a set homeomorphic to X, if necesssary, we may assume that $\mathbf{0} \in \overset{o}{X}$ and that $\mathbb{D}^n \subset X$.

Define $f : \mathbb{S}^n \to [1, \infty)$ by $f(x) = \sup\{t \,|\, tx \in X\}$. Since X is compact, and therefore bounded and closed, and convex, it is clear that f is well defined, and that $tx \in X$ when $t \in [0, f(x)]$. We will show that f is continuous. Let $x_0 \in \mathbb{S}^n$ and let x_k be a sequence in $\mathbb{S}^n$ with limit x_0. We must show that $f(x_k) \to f(x_0)$. Since X, and therefore f is bounded we may, by replacing x_k by some subsequence, assume that $f(x_k)$ is convergent to some t_0, and we must prove that $t_0 = f(x_0)$.

To this end, let $u_1, \dots u_n$ be a basis of $\mathbb{R}^n$ with $u_i \in \mathbb{S}^n$ for each i, and $u_1 = x_0$. For each k, we can write $x_k = \sum_{i=1}^n a_i(k) u_i$. Define $v_i(k) = -u_i$ if $a_i(k) < 0$ and let $v_i(k) = u_i$ otherwise. Redefining the constants $a_i(k)$ if necessary, we can write $x_k = \sum_{i=1}^n a_i(k) v_i(k)$ with $a_i(k) \geq 0$ for each i and k, and we get that $\sum_{i=0}^n \frac{a_i(k)}{f(v_i(k))} > 0$. Put $t(k) = (1/\sum_{i=0}^n \frac{a_i(k)}{f(v_i(k))})$ and $b_i(k) = \frac{t(k) a_i(k)}{f(v_i(k))}$. Then

$$\sum_{i=1}^n b_i(k) f(v_i(k)) v_i(k) = t(k) \sum_{i=1}^n a_i(k) v_i(k) = t(k) x_k.$$

Since $\sum_{i=1}^n b_i(k) = 1$, $b_i(k) \geq 0$, $f(v_i(k)) v_i(k) \in X$ and X is convex, we get that $t(k) x_k \in X$. From the definition of f, it follows that $f(x_k) \geq t(k)$.

Since $x_k \to x_0$, we must have that $a_1(k) \to 1$ and $a_i(k) \to 0$ for $i > 1$ as $k \to \infty$, since it follows from our definitions that $x_k \to x_0$ implies that $v_1(k) = u_1 = x_0$ for k large. It follows that $t(k) \to f(x_0)$. Since $f(x_k) \geq t(k)$, we get that $t_0 \geq f(x_0)$. On the other hand, since $f(x_k) x_k \in X$, X is closed and $f(x_k) x_k \to t_0 x_0$, we get that $t_0 x_0 \in X$. So $t_0 > f(x_0)$ is impossible (by the definition of f), and we must have $t_0 = f(x_0)$. The continuity of f follows from this.

We will now see that $\partial X = \{f(x) x \,|\, x \in \mathbb{S}^n\}$. Since $tx \notin X$ for $t > f(x)$, and $tx \in X$ for $t \in [0, f(x)]$ it is clear that $\{f(x) x \,|\, x \in \mathbb{S}^n\} \subset \partial X$. If $0 < t < f(x)$, it follows from the continuity of f that there exists an open neighborhood V of x in $\mathbb{S}^n$ such that $f(y) > \frac{t}{2} + \frac{f(x)}{2}$ when $y \in V$. $\{sy \,|\, s \in (0, \frac{t}{2} + \frac{f(x)}{2}),\, y \in V\}$ is thus an open neighborhood of tx contained in X, and we get that $tx \in \overset{0}{X}$. $\partial X = \{f(x) x \,|\, x \in \mathbb{S}^n\}$ follows from this.

Let $g : X \to \mathbb{R}^n$ be defined by $g(x) = x / f(\frac{x}{||x||})$, when $x \neq 0$ and $f(\mathbf{0}) = \mathbf{0}$. Since f is continuous and ≥ 1, g is continuous. We it leave as an easy exercise to prove that f is one-to-one with image equal $\mathbb{D}^n$ and that $f(\partial X) = f(\{f(x) x \,|\, x \in \mathbb{S}^n\}) = \mathbb{S}^n$. Since X is compact and $\mathbb{D}^n$ is Hausdorff, it follows that f also is closed and therefore a homemorphism between $(X, \partial X)$ and $(\mathbb{D}^n, \mathbb{S}^n)$.