Homework 1 - Algebraic Topology, I.

Exercise 1. (Optional) Student expectations survey: Send an e-mail to your instructor. Include some information about your mathematical background, expectations, and special requests for this class.

Exercise 2. The following pairs of topological spaces are homeomorphic:

a) The circle S^1 and the quotient space $\mathbb{R}/\mathbb{Z}$.
b) The torus $I \times I/\sim$ and $S^1 \times S^1$ where $(0,t) \sim (1,t)$ and $(t,0) \sim (t,1)$.
c) The real projective space $\mathbb{RP}^n$ and $S^n/\sim$ where $x \sim -x$.
d) X and $X \wedge S^0$ for any topological space X.
e) The unreduced suspension of S^n and S^{n+1}.

Exercise 3. Take as our alphabet:

A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, Z.

Divide the letters into classes of homotopy equivalent letters.

Exercise 4. Suppose $A \subset \mathbb{R}^n$ is convex and $a \in A$. Show that $\{a\}$ is a deformation retract of A.

Exercise 5. From chapter 0 in Hatcher's book: 1, 2, 3, 4, 5, 6.

Exercise 6. Show that X is contractible if and only if every map with X as source or target is homotopic to a constant.

Exercise 7. Suppose $X \subseteq \mathbb{R}^n$ is compact, convex with a nonempty interior. Then the pair $(X, \partial X)$ is homeomorphic to (D^n, S^{n-1}). (In particular, $(I^n, \partial I^n)$ is homeomorphic to (D^n, S^{n-1}).)

Exercise 8. Identify the pushouts of the following diagrams:

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & D^n \\ \downarrow & & \\ D^n & & \end{array} \qquad \begin{array}{ccc} S^{n-1} & \longrightarrow & * \\ \downarrow & & \\ * & & \end{array}$$

Exercise 9. Equip $M_n(\mathbb{R})$ with the metric

$$d(A,B) = \sqrt{\sum_{i,j=1}^{n} (a_{ij} - b_{ij})^2}.$$

With this topology $M_n(\mathbb{R})$ is homeomorphic to $\mathbb{R}^{n^2}$. Consider $GL_n(\mathbb{R})$ as a subspace of $M_n(\mathbb{R})$. Identify the path-components $\pi_0 GL_n(\mathbb{R})$ of $GL_n(\mathbb{R})$. Show that $SL_n(\mathbb{R})$ is a deformation retract of $GL_n^+(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) | \det(A) > 0\}$.

Exercise 10. The following pairs of topological spaces are homeomorphic:

a) For X pointed, C^*X/X and the reduced suspension ΣX.
b) $\Sigma I^{n-1}/\partial I^{n-1}$ and $I^n/\partial I^n$.
c) ΣS^{n-1} and S^n.
d) S^n and $S^1 \wedge \cdots \wedge S^1$.
e) I^n and D^n.
f) $I^n \setminus \partial I^n$ and $\mathbb{R}^n$.
g) $S^n \setminus \{(1, 0, \ldots, 0)\}$ and $\mathbb{R}^n$.