Homework 2 - Algebraic Topology, I.

Exercise 1. **a**) The cube

$$I^n \equiv \{(x_1, \ldots, x_n) \in \mathbb{R}^n | 0 \leq x_i \leq 1, 1 \leq i \leq n\}$$

is contractible. It might be a good idea to start out with the special case $n = 1$.

b) For $f\colon X \to Y$, show that (M_f, Y) satisfies HEP.

c) By contemplating the squaring map $S^1 \to S^1$ given by $z \mapsto z^2$, show that $\mathbb{RP}^1$ is homeomorphic to the circle. Identify the mapping cone and mapping cylinder of the squaring map on S^1.

d) Suppose $f\colon X \to S^n$ is a map which is not surjective. Prove that f is homotopic to a constant map.

e) Recall the cone construction

$$CX \equiv \frac{X \times I}{X \times \{0\}}.$$

Explicate a homotopy between the constant map with value $[x, 0]$ and id_{CX}.

Exercise 2. Recall - from class - that $i\colon A \to X$ is a cofibration if for all spaces Y and maps $h\colon A \times I \to Y$, $f\colon X \to Y$ such that $h(a, 0) = f(i(a))$ for all $a \in A$, there exists a map $H\colon X \times I \to Y$ - the big brother of h - such that $H(x, 0) = f(x)$ and $h(a, t) = H(i(a), t)$ for all $a \in A$, $x \in X$, and $t \in I$.

Now let $f\colon X \to Y$ be any map and $X \to M_f$ the embedding of X - given by $x \mapsto [x, 0]$ - into the mapping cylinder of f. In this exercise we complete a proof given in class showing that $X \to M_f$ is a cofibration. What remains to be proven are the following statements.

a) The inclusion $\partial I \subseteq I$ is a cofibration.

b) If $i\colon A \to X$ is a cofibration, then for all Y the map $i \times \mathrm{id}_Y\colon A \times Y \to X \times Y$ is a cofibration.

c) The composite of two cofibrations is a cofibration.

Exercise 3. From chapter 0 in Hatcher's book: 9, 10, 11, 14, 16, 17, 18, 19, 23, 24.

Exercise 4. **a**) Read §1-4 of Chapter 2 in "A concise course in algebraic topology."

Exercise 8 in Homework 1 shows that pushouts do not exist in the homotopy category of topological spaces. Next we define "homotopy invariant" pushouts called "homotopy pushouts" as double mapping cylinders. The homotopy pushout of the diagram

$$Z \xleftarrow{g} X \xrightarrow{f} Y$$

is the quotient space

$$P \equiv Y \coprod (X \times I) \coprod Z / \sim$$

where

$$(x, 0) \sim f(x), \quad (x, 1) \sim g(x).$$

b) Show that the homotopy pushout of the diagram

$$X \longleftarrow X \times Y \longrightarrow Y$$

is the join $X \star Y$ defined as the pushout of

$$X \times \partial I \subset X \times I \times Y$$

and the map

$$X \times \partial I \times Y \to X \coprod Y$$

formed from the projections

$$X \times \{0\} \times Y \to X$$

and

$$X \times \{1\} \times Y \to Y.$$

Identify the homotopy pushout of the diagram

$$* \longleftarrow X \longrightarrow *$$

as the suspension of X.

c) Show that the homotopy pushouts of the diagrams

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & D^n \\ \downarrow & & \\ D^n & & \end{array} \qquad \begin{array}{ccc} S^{n-1} & \longrightarrow & * \\ \downarrow & & \\ * & & \end{array}$$

are homotopy equivalent to S^n.

Exercise 5. a) Read §6 of Chapter 2 in "A concise course in algebraic topology."