THE SNAKE LEMMA AND THE LONG EXACT SEQUENCE

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1. The snake lemma

Lemma 1.1. *Suppose that*

$$\begin{array}{ccccccccc}
 & & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\
 & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow & & \\
0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & &
\end{array}$$

is a commutative diagram, with exact rows. For $c \in \ker(\gamma)$ let $s(c) = a' + \mathrm{im}(\alpha) \in \mathrm{cok}(\alpha)$, where $g(b) = c$ and $f'(a') = \beta(b)$. Then

$$\ker(\alpha) \xrightarrow{f} \ker(\beta) \xrightarrow{g} \ker(\gamma) \xrightarrow{s} \mathrm{cok}(\alpha) \xrightarrow{f'} \mathrm{cok}(\beta) \xrightarrow{g'} \mathrm{cok}(\gamma)$$

is exact. If $f\colon A \to B$ is injective and $g'\colon B' \to C'$ is surjective, then

$$0 \to \ker(\alpha) \xrightarrow{f} \ker(\beta) \xrightarrow{g} \ker(\gamma) \xrightarrow{s} \mathrm{cok}(\alpha) \xrightarrow{f'} \mathrm{cok}(\beta) \xrightarrow{g'} \mathrm{cok}(\gamma) \to 0$$

is exact.

2. The long exact sequence

Lemma 2.1. *Let*

$$0 \to A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \to 0$$

be a short exact sequence of chain complexes. Then

$$\cdots \to H_{n+1}(C_*) \xrightarrow{\partial} H_n(A_*) \xrightarrow{f_*} H_n(B_*) \xrightarrow{g_*} H_n(C_*) \xrightarrow{\partial} H_{n-1}(B_*) \to \dots$$

is a long exact sequence.

Proof. From the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n & \longrightarrow & 0 \\
 & & {\scriptstyle\partial_n}\downarrow & & {\scriptstyle\partial_n}\downarrow & & {\scriptstyle\partial_n}\downarrow & & \\
0 & \longrightarrow & A_{n-1} & \xrightarrow{f_{n-1}} & B_{n-1} & \xrightarrow{g_{n-1}} & C_{n-1} & \longrightarrow & 0
\end{array}$$

with exact rows, and the snake lemma, we get an exact sequence

$$0 \to Z_n(A_*) \xrightarrow{f_n} Z_n(B_*) \xrightarrow{g_n} Z_n(C_*) \xrightarrow{\partial} A_{n-1}/B_{n-1}(A_*) \xrightarrow{f_{n-1}} B_{n-1}/B_{n-1}(B_*) \xrightarrow{g_{n-1}} C_{n-1}/B_{n-1}(C_*) \to 0\,.$$

Hence we have a commutative diagram

$$\begin{array}{ccccccccc}
 & & A_n/B_n(A_*) & \xrightarrow{f_n} & B_n/B_n(B_*) & \xrightarrow{g_n} & C_n/B_n(C_*) & \longrightarrow & 0 \\
 & & {\scriptstyle\partial_n}\downarrow & & {\scriptstyle\partial_n}\downarrow & & {\scriptstyle\partial_n}\downarrow & & \\
0 & \longrightarrow & Z_{n-1}(A_*) & \xrightarrow{f_{n-1}} & Z_{n-1}(B_*) & \xrightarrow{g_{n-1}} & Z_{n-1}(C_*) & &
\end{array}$$

with exact rows. Applying the snake lemma again, we get an exact sequence

$$H_n(A_*) \xrightarrow{f_*} H_n(B_*) \xrightarrow{g_*} H_n(C_*) \xrightarrow{\partial} H_{n-1}(A_*) \xrightarrow{f_*} H_{n-1}(B_*) \xrightarrow{g_*} H_{n-1}(C_*)\,.$$

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