Def. [Cell Complex]

(1) Start with a discrete set X^0 of points (0-cells)

(2) Inductively:

Define maps $\varphi_\alpha : S^{n-1} \to X^{n-1}$

n = dimension of cells. $\longrightarrow$ Form $X^n = X^{n-1} \sqcup_\alpha D^n_\alpha / x \sim \varphi_\alpha(x)$ for $x \in \partial D^n_\alpha$

(3) $X = X^n$ for some $n < \infty$ or $X = \bigcup_n X^n$ with weak topology: $\underset{\text{open}}{A \subset X} \iff \underset{\text{open}}{A \cap X^n \subset X^n}$ $\forall n = 0, 1, \ldots$

(2) $\Rightarrow$ $X^n = X^{n-1} \sqcup_\alpha e^n_\alpha$ $\quad e^n_\alpha$ is an open n-disk

Each cell has its characteristic map. $\Phi_\alpha : D^n_\alpha \hookrightarrow X^{n-1} \sqcup_\alpha D^n_\alpha \to X^n \hookrightarrow X$

$\Phi_\alpha |_{\text{int} D_\alpha}$ homeomorphic onto e^n_α

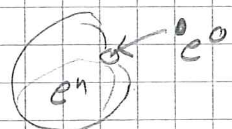
Why CW?

1) Closure-finitness: The closure of a cell meets only finitely many other cells

2) Weak topology: A set is closed iff it meets the closure of each cell in a closed set.

Ex.

- 1-dim cell complex: Graph.
- $S^n = \{e^0, e^n\}$



n-cell: $S^{n-1} \to e^0$

$S^n = D^n / \partial D^n$.

- $\mathbb{R}P^n = e^0 \cup e^1 \cup \cdots \cup e^n$

$$\mathbb{R}P^n = S^n / v \sim -v = D^n / \underbrace{v \sim -v, v \in \partial D^n}_{\mathbb{R}P^{n-1}}$$

$$\Rightarrow \mathbb{R}P^n = \mathbb{R}P^{n-1} \sqcup e^n / S^{n-1} \to \mathbb{R}P^{n-1}$$

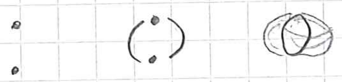
$A \subset X$ ← Cell complex
closed union of cells

$\Rightarrow$ A is itself a cell complex

$\operatorname{Im} \Phi_\alpha \subset A$. if α is a cell in A

(X,A) is called a CW-pair

Ex. $S^0 \subset S^1 \subset S^2 \subset \dots$ as cell complexes



Attach two cells each time.

Products: X, Y cell complexes $\Rightarrow$ $X \times Y$ cell complex

Cells: $e^m_\alpha \times e^n_\beta$

Problem: topology of $X \times Y$ may be finer than the product topology

No problem if X and Y have countable no. of cells

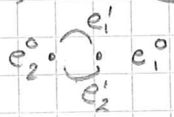
e^m_α : open m-disc.

$\cong \{(x_1, \dots, x_m) \mid |x_i| < 1\}$ $\Rightarrow$ Product ok.

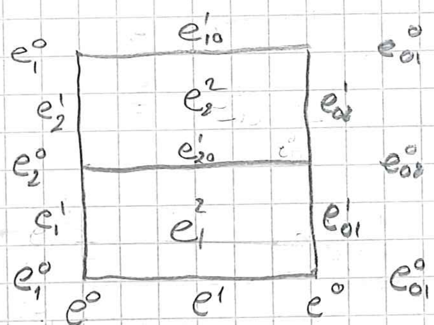
Ex. Torus $T = S^1 \times S^1$

Cell structure of S^1: $\{e^0, e^1\}$

Alternative $\{e^0_1, e^0_2, e^1_1, e^1_2\}$



Torus: $\{e^0 \times e^0_1, e^0 \times e^0_2, e^0 \times e^1_1, e^0 \times e^1_2, e^1 \times e^0_1, e^1 \times e^0_2, e^1 \times e^1_1, e^1 \times e^1_2\}$
$= \{e^0_{01}, e^0_{02}, e^1_{01}, e^1_{02}, e^1_{10}, e^1_{20}, e^2_1, e^2_2\}$



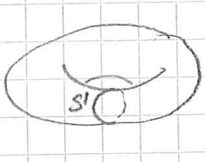
Quotients: (X,A) CW pair

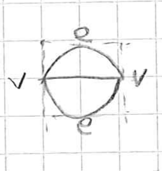
X/A cell complex structure

Cells: $X \setminus A$, $A \rightsquigarrow$ new 0-cell

e^n_α : attached by $\varphi_\alpha : S^{n-1} \to X^{n-1} \to X^{n-1}/A^{n-1}$

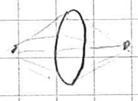
Ex. (T, S^1)





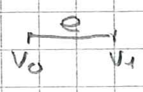
Suspension $SX = X \times I / _{X\times\{0\},\, X\times\{1\}}$

Ex.

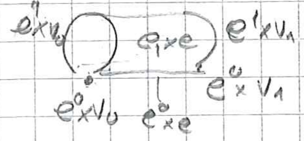


$S^1 = \{e^0, e^1\}$

$I = \{v_0, v_1, e\}$



$S^1 \times I = \{ e^0\times v_0,\ e^0\times v_1,\ e^1\times v_0,\ e^1\times v_1,\ e^0\times e,\ e^1\times e \}$

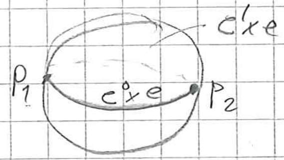


$S^1\times\{0\} = \{e^0\times v_0, e^1\times v_0\}$

$S^1\times\{1\} = \{e^0\times v_1, e^1\times v_1\}$

$S^1\times I / _{S^1\times\{0\},\, S^1\times\{1\}}$

$= \{ \underset{S^1\times\{0\}}{\overset{\shortparallel}{P_1}},\ \underset{S^1\times\{1\}}{\overset{\shortparallel}{P_2}},\ e^0\times e,\ e^1\times e \}$



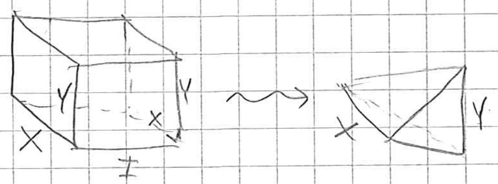
Join Generalize cone: $X\times I / _{X\times\{0\}}$

suspension: $X\times I / _{X\times\{0\},\, X\times\{1\}}$

Join $X\times Y\times I / _{\{x\}\times Y\times\{0\},\ \forall x \text{ and } X\times\{y\}\times\{1\}\ \forall y}$

i.e. $X\times Y\times\{0\} \underset{\text{collaps to}}{\rightsquigarrow} X$

$X\times Y\times\{1\} \underset{\text{collaps to}}{\rightsquigarrow} Y$



Important example: $\underbrace{\{pt\} * \{pt\} * \cdots * \{pt\}}_{n} = \Delta^{n-1}$ n-simplex

Another example: $\underbrace{S^0 * S^0 * \cdots * S^0}_{n} = S^{n-1}$ sphere.

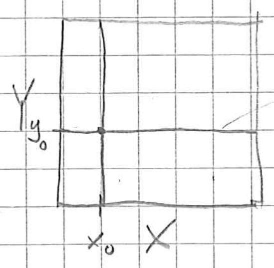
Wedge sum: $(X, x_0 \in X)$ $(Y, y_0 \in Y)$

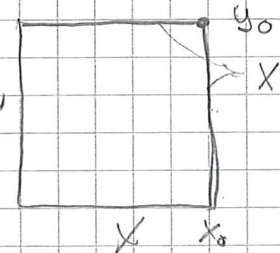
$$X \vee Y = X \sqcup Y / x_0 \sim y_0$$

Ex. $S^1 \vee S^1 =$  Cell structure; obvious

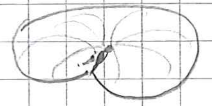
Ex. $X^n / X^{n-1} = \bigvee_\alpha S^n_\alpha$

Smash product: $X \wedge Y = X \times Y / X \vee Y$





Ex $S^1 \times S^1$ 

 $\simeq S^2$

$S^m = \{e^0, e^m\}$

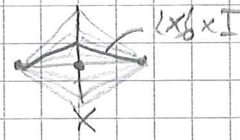
$S^n \times S^m = \{\underbrace{e^0, e^n, e^m}_{\to e^0}, e^{n+m}\}$, $S^m \vee S^n = \{e^0, e^m, e^n\}$

$S^n \times S^m / S^m \vee S^n = \{e^0, e^{n+m}\} = S^{n+m}$

Reduced suspension:

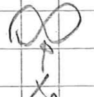
$x_0 \in X$ CW-complex
0-cell

$\{x_0\} \times I \subset SX$



$\Sigma X = SX / \{x_0\} \times I \simeq SX$
homotopy equiv.

Ex $X = S^1 \vee S^1$



SX 

$S^2 \cup S^2$, $S^2 \cap S^2 = \{x_0\} \times I$

$\Rightarrow \Sigma X = S^2 \vee S^2$

In general 

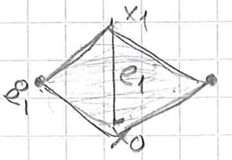
$\Sigma(X \vee Y) = \Sigma X \vee \Sigma Y$

as cell complexes

$$SX = \{e_1^0, e_2^0\} \cup \{e_j^n \times (0,1) \mid e_j^n \in X^n\} = X\times I / X \times \partial I$$

$$\Sigma X = \{x_0\} \cup \{e_j^{n+1} \mid e_j \in X^n, e_j \neq x_0\} = SX / \{x_0\}\times I$$

SX: e_1^0 x_1 e_1 x_0 e_2^0



ΣX: x_1 $\{x_0\}\times I$

Alt:

$$X\times I$$

$$\swarrow \qquad \searrow$$

$$CX = X\times I / X\times\{0\} \qquad X\times I/\partial I = X \times S^1$$

$$SX = (X\times I / X\times\{0\}) / X \times \{1\} \qquad X\times S^1 / X\vee S^1 = X \wedge S^1$$

$$\downarrow$$

$$\Sigma X = SX / x_0 \times I$$

Collapsed? ΣX: $X\times\{0\} \to pt$ $\quad X\times\{1\} \to pt$ $\quad x_0\times I \to X\times\{0\} \cup X\times\{1\} \cup \{x_0\}\times I$

$= (X\times\partial I) \cup (\{x_0\}\times I)$

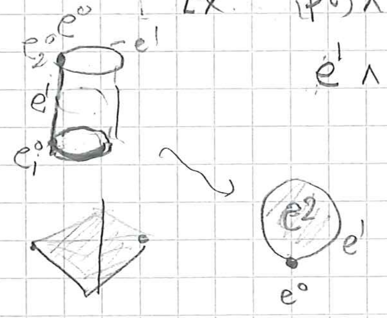
$X\wedge S^1$ $\quad X\times\partial I \to pt$ $\quad X\vee S^1 = (X\times\underbrace{\{s_0\}}_{=\partial I}) \cup (\{x_0\}\times S^1) \to pt$

$= (X\times\partial I) \cup (\{x_0\}\times I)$.

Ex. $\{pt\}\wedge S^1 = \{pt\}$ $\qquad \Sigma\{pt\} = \{x_0\}\times I / \{x_0\}\times I$

$e^1\wedge S^1 = \{e_1^0, e_2^0, e^1\}\times\{e^0, e^1\} / e^1 \vee S^1$

$= \{v_1, v_2, e_1^1, e_2^1, e_0^1, e^2\} / \{v_1, v_2, e_0^1, e_1^1\} = \{e^0, e^1, e^2\}$

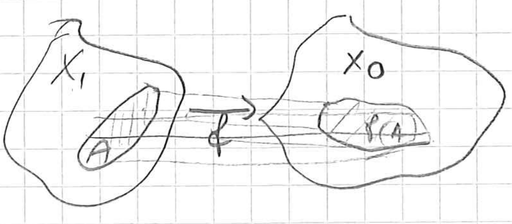


Attached space

$$\begin{array}{ccc} X_1 & & X_0 \\ \cup\!\mid & \nearrow f & \\ A & & \end{array}$$

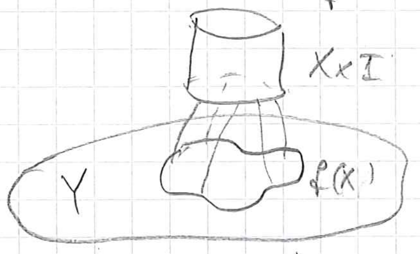
$$X_0 \amalg X_1 / a\sim f(a) = X_0 \sqcup_f X_1$$

X_0 with X_1 attached along A.

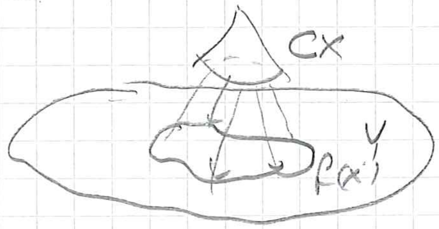


Mapping cylinder: $\quad M_f = Y \sqcup_f (X \times I)$ $\qquad f : X \times \{0\} \to Y$

$f: X \to Y$



Mapping cone:



<u>Prop.</u> (X_1, A) CW-pair, $f \simeq g : A \to X_0$ homotopic $\Rightarrow X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1$

Proof. ($W \simeq Z$ rel Y for pairs (W, Y) and (Z, Y)

means: $\exists\ \varphi : W \to Z$, $\psi : Z \to W$ such that

$$\varphi|_Y = \psi|_Y = \mathbb{1}_Y$$

and $\varphi\psi \simeq \mathbb{1}_Z$, $\psi\varphi \simeq \mathbb{1}_W$.)

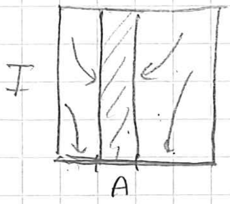
Let $F : A \times I \to X_0$ be a homotopy $f \simeq g$.

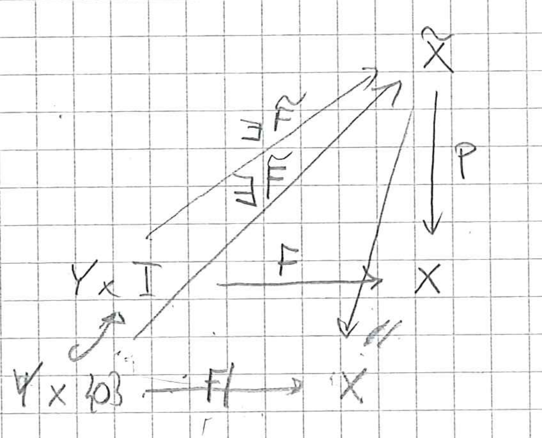
$$X_0 \sqcup_F (X_1 \times I)$$

$$\swarrow \simeq \qquad \searrow \simeq$$

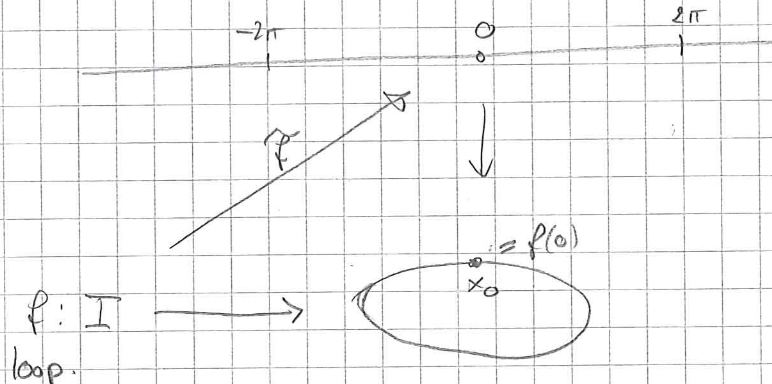
$$X_0 \sqcup_f X_1 \qquad\qquad X_0 \sqcup_g X_1$$

Induced by $X_1 \times I \longrightarrow X_1 \times \{0\} \cup A \times I$





Special cases $Y = \{pt\}$ or $Y = I$



$\tilde{f}(0) = 0 \quad \tilde{f}(1) = n 2\pi$

$\simeq \tilde{\omega}_n$, $\omega_n(s) = (\cos 2\pi n s, \sin 2\pi n s)$

$\omega_n \simeq \omega_m$ via f_t $\rightsquigarrow$ lifts to $\tilde{f}_t$ uniqueness $\Rightarrow \tilde{\omega}_n \simeq \tilde{\omega}_m$

But $\tilde{f}_t(1)$ constant

$\Rightarrow \underset{\tilde{f}_0(1)}{\underset{\parallel}{\tilde{\omega}_n(1)}} = \underset{\tilde{f}_1(1)}{\underset{\parallel}{\tilde{\omega}_m(1)}}$

$\Rightarrow \quad n = m$.