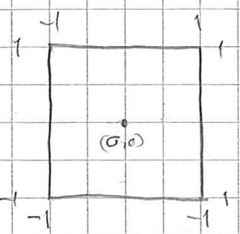
1.



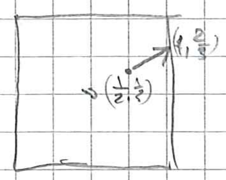
$$f_t(x,y) = \begin{cases} (1-t)(x,y) + \frac{t}{|x|}(x,y) & |y| \le |x| \\ (1-t)(x,y) + \frac{t}{|y|}(x,y) & |x| \le |y| \end{cases}$$

Eks $(x,y) = (\frac{1}{2}, \frac{1}{3})$: $f_t(\frac{1}{2}, \frac{1}{3}) = (1-t)(\frac{1}{2}, \frac{1}{3}) + \frac{t}{\frac{1}{2}}(\frac{1}{2}, \frac{1}{3})$

$$= (1-t+2t)(\tfrac{1}{2}, \tfrac{1}{3}) = (1+t)(\tfrac{1}{2}, \tfrac{1}{3})$$

$t=0$: $f_0(\frac{1}{2}, \frac{1}{3}) = (\frac{1}{2}, \frac{1}{3})$

$t=1$: $f_1(\frac{1}{2}, \frac{1}{3}) = (1, \frac{2}{3})$



2. $\mathbb{R}^n \setminus \{0\} \twoheadrightarrow S^{n-1}$

$$f_t(x_1,\dots,x_n) = (1-t)(x_1,\dots,x_n) + \frac{t}{\sqrt{x_1^2+\dots+x_n^2}}(x_1,\dots,x_n)$$

$$= \left(1-t+\frac{t}{\sqrt{x_1^2+\dots+x_n^2}}\right)(x_1,\dots,x_n)$$

3.

a) $X \underset{\exists\varphi}{\overset{f}{\rightleftarrows}} Y \underset{\exists\psi}{\overset{g}{\rightleftarrows}} Z$ $\qquad \varphi f \simeq 1\!\!1_X$, $f\varphi \simeq 1\!\!1_Y \simeq \psi g$, $g\psi \simeq 1\!\!1_Z$

$$\Rightarrow \varphi\psi g f \simeq \varphi 1\!\!1_Y f \simeq \varphi f \simeq 1\!\!1_X$$

$$g f \varphi \psi \simeq g 1\!\!1_Y \psi \simeq g\psi \simeq 1\!\!1_Z$$

Transitive

Reflexive: $X \xrightarrow{id} X$ ok

Symmetric: $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ ok

b) $f_0, f_1 : X \to Y$, $f_0 \simeq f_1$ if $\exists$ $f_t : X \to Y$ which restricts, and continuous

R: $f \simeq f$ $\quad f_t = f$ constant homotopy

S: $f_0 \simeq f_1 \Rightarrow f_1 \simeq f_0$ via f_{1-t}

T: $f_0 \underset{\varphi_t}{\simeq} f_1$, $f_1 \underset{\psi_t}{\simeq} f_2$ $\qquad h_t = \begin{cases} \varphi(2t) & [0,\frac{1}{2}] \\ \psi(2t-1) & [\frac{1}{2},1] \end{cases}$

c) $f : X \to Y$, $f \simeq g$ via φ_t, $\exists h : Y \to X$ s.t. $gh \simeq 1\!\!1_Y$, $hg \simeq 1\!\!1_X$

We have: $hf \simeq hg \simeq 1\!\!1_X$ and $fh \simeq gh \simeq 1\!\!1_Y$.

5. Deformation retraction of X to a point $x_0 \in X$:

$$f_t : X \to X$$

such that $f_1 = 1\!\!1_X$, $f_0(X) = x_0$, $f_t(x_0) = x_0 \;\forall t \in I$.

Let $U \subseteq X$ neighbourhood of x_0

Let $V = \bigcap_{t \in I} f_t^{-1}(U)$. Then $V \subset f_t^{-1}(U)$ $\forall t$, and $f_t(V) \subseteq U$. Since $f_t(x_0) = x_0$ it follows that $x_0 \in V \neq \emptyset$.

and also, because $f_1^{-1}(U) = 1\!\!1_1^{-1}(U) = U$, $V \subseteq f_1^{-1}(U) = U$.

Denote by $\iota : V \to U$, the inclusion.

We define a homotopy $g_t = f_t|_V \circ \iota : V \to U$ where

$g_0 = f_0|_V \circ \iota = x_0$ and $g_1 = f_1|_V \circ \iota = 1\!\!1_V \circ \iota = \iota : V \to U$.

Some problems!

5. $\forall \in X$ $\left(f_t = (1-t)\mathbb{1}_X + t\cdot x_0\right)$ $\quad f_t : X \to X$, $f_1 = \mathbb{1}_X$, $f_0 = x_0$

$x \in U \subset X$ $\quad f_t(x_0) = x_0 \ \forall t$.

neighbourhood

Suppose $f_t^{-1}(U) \subseteq U$ for all $t \leq T$. By continuity this is, the. Let

$V = f_T^{-1}(U) \hookrightarrow U$. T

9. Contractible space : Homotopy type of point $\quad x_0 \in X$, $\varphi_t : X \to X$

s.t. $\varphi_0 = \mathbb{1}_X$, $\varphi_1 = x_0 \in A$.

Retract : $r : X \to X$, $r(X) = A$, $r^2 = r$

$\Rightarrow r\varphi_t|_A : A \to A$ continuous such that

$r\varphi_0|_A = r\,\mathbb{1}_X|_A = r\,\mathbb{1}_A = r|_A = \mathbb{1}_A$

$r\varphi_1|_A = r(x_0) = x_0$

10.
1) X contractible $\iff \forall f : X \to Y$, $f \simeq y_0$ (some pt.)

Proof $\Rightarrow$. $\exists$ homotopy $\varphi_t : X \to X$ and $x_0 \in X$ s.t. $\varphi_0 \simeq \mathbb{1}_X$, $\varphi_1 = x_0$

Let $f : X \to Y$, then $\psi_t = f\varphi_t : X \to Y$, continuous

$\psi_0 \simeq f\varphi_0 \simeq f \circ \mathbb{1}_X = f$

$\psi_1 \simeq f\varphi_1 \simeq f(x_0) \doteq y_0 \in Y$.

$\Leftarrow$ Apply to $id : X \to X$

2) X contractible $\iff \forall f : Y \to X$, $f \simeq x_0$

$\Leftarrow$ Apply to $id : X \to X$.

$\Rightarrow$. $\varphi_t : X \to X$ continu and $\exists\, x_0 \in X$ s.t. $\varphi_0 \simeq \mathbb{1}_X$, $\varphi_1 \simeq x_0$

For any $f : Y \to X$ put $\psi_t = \varphi_t f : Y \to X$ (continous)

$\psi_0 = \varphi_0 f \simeq \mathbb{1}_X f = f$

$\psi_1 = \varphi_1 f \simeq x_0 f = x_0$.

11.

a) $\exists\, g,h: Y \to X$ s.t. $fg \simeq 1\!\!1_Y$ and $hf \simeq 1\!\!1_X$ $\Rightarrow$ $f: X \to Y$ homotopy equiv.

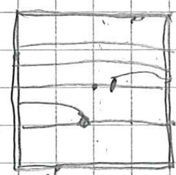
Proof: We have $g \simeq hfg \simeq h$. Thus $gf \simeq hf \simeq 1\!\!1_X$. So $g: Y \to X$ gives a homotopy inverse.

b) Suppose fg and hf are homotopy equiv. with $p: Y \to Y$, $q: X \to X$ as homotopy inverses. Then f is a homotopy equivalence.

Proof: We have $f(gp) \simeq (fg)p \simeq 1\!\!1_Y$ and $q(hf) \simeq (qh)f \simeq 1\!\!1$. Thus by a), f is a homotopy equivalence.

14. $v - e + f = 2$

S^2: Since $v > 0$ we have $\exists\, v_1 \in S^2$...



$f = n+1$ 2-cells

$f - e + v = n+1-n+1$ $\Leftarrow$ $e = n$ edges

$= \underline{2}$ $\quad v = 1$ vertex

Insert $v-1$ new vertices, draw edges to the basepoint $\Rightarrow f - ((f-1)+(v-1)) + 1 + v - 1$

$= \cancel{f} - \cancel{f} + 1 - \cancel{v} + 1 + \cancel{1} + \cancel{v} - \cancel{1} = 2$

18.

$S^1 * S^1 = S^3$

$u \cdot (x_1, x_2, 0, 0) + (1-u)(0,0,y_1,y_2) = (ux_1, ux_2, (1-u)y_1, (1-u)y_2)$

$\mapsto (ux_1, ux_2, (1-u)y_1, (1-u)y_2) \dfrac{1}{\sqrt{u^2x_1^2 + u^2x_2^2 + (1-u)^2y_1^2 + (1-u)^2y_2^2}}$

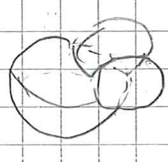
$\sqrt{} = \sqrt{u^2 + (1-u)^2} = \sqrt{1 - 2u + 2u^2} \neq 0.$

since $u^2 + (1-u)^2 = 0 \Leftrightarrow u = 1-u = 0$ impossible.

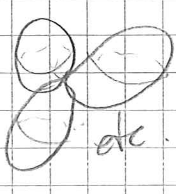
General $S^m * S^n = S^{m+n-1}$

$u(\underline{x}, 0) + (1-u)(0, \underline{y}) \mapsto \dfrac{\underline{x}, \underline{y}}{\sqrt{u^2 + (1-u)^2}}$ $\quad \underline{\text{ok}}$

19.



etc. $\rightsquigarrow$



etc.