$$\begin{array}{ccc} \varphi : X & \to & Y \\ \Psi & & \Psi \\ x_0 & \mapsto & y_0 \end{array} \quad ; \quad \varphi : (X, x_0) \longrightarrow (Y, y_0)$$

Induces homomorphism $\varphi_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$

as follows.:

Pf

Let $f : I \to X$ loop, $f(0) = f(1) = x_0$

$\Rightarrow \quad \varphi f : I \to Y$ loop $\varphi f(0) = \varphi(x_0) = y_0$

$\varphi f(1) = \varphi(x_0) = y_0$

$f_0 \overset{f_t}{\simeq} f_1 \Rightarrow \varphi f_0 \overset{\varphi f_t}{\simeq} \varphi f_1$

$\rightsquigarrow \quad \varphi_* [f] = [\varphi f]$ well-defined

Furthemore

$$\varphi \circ (f \cdot g) = \varphi f \cdot \varphi g$$

since $f \cdot g(s) = \begin{cases} f(2s) & 0 \le s \le \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}$

$$\Rightarrow \varphi \circ (f \cdot g)(s) = \begin{cases} \varphi f(2s) \\ \varphi g(2s-1) \end{cases} = (\varphi f) \cdot (\varphi g)(s)$$

Some properties:

1) $(\varphi \circ \psi)_* = \varphi_* \circ \psi_*$ $\quad (X, x_0) \xrightarrow{\psi} (Y, y_0) \xrightarrow{\varphi} (Z, z_0)$

Because $(\varphi \psi)_* [f] = [\varphi \psi f] = \varphi_* [\psi f] = \varphi_* \psi_* [f]$.

2) $\mathbb{1}_{X*} = \mathbb{1}_{\pi_1}$

Because $\mathbb{1}_* [f] = [\mathbb{1} \circ f] = [f]$

$\Longrightarrow \quad \pi_1$: Category of pointed top. spaces $\longrightarrow$ Category of groups

Notice that if φ is a homeomorphism s.t. $\varphi \psi = \mathbb{1}$

then $\varphi_* \psi_* = \mathbb{1}_{\pi_1}$ (iso of groups)

Lemma

$X = \bigcup_\alpha A_\alpha$ (open cover), $x_0 \in A_\alpha \ \forall \alpha$, $A_\alpha \cap A_\beta$ path-connected, $[f] \in \pi_1(X, x_0)$

$\Rightarrow [f] = [f_1]\cdot[f_2]\cdots[f_m]$, $[f_\alpha] \in \pi_1(A_\alpha, x_0)$

Proof $f: I \to X$, $f(0) = f(1) = x_0$

$\exists$ partition $0 = s_0 < s_1 < \cdots < s_m = 1$ s.t. $f([s_i, s_{i+1}]) \subset A_\alpha$ for some α

$\forall s \in I$, $f(s) \in A_\alpha \Rightarrow s \in V_s$ (open in I) ; $f(V) \subseteq A_\alpha$ by continuity and openness of A_α

In fact $s \in W_s \subset \overline{W_s} \subset V_s$ (W_s open)

and $\bigcup_s W_s = X$ By compactness of I, finite is enough

$\rightsquigarrow \overline{W_s} = [s_i, s_{i+1}]$

$f([s_{i-1}, s_i]) \subset A_\alpha$ we rename to A_i

Let $f_i = f|_{A_i} \Rightarrow f = f_1 \cdots f_m$

$A_i \cap A_{i+1}$ is path-connected with $g_i : I \to A_i \cap A_{i+1}$, $g_i(0) = x_0$, $g_i(1) = f(s_i) \in A_i \cap A_{i+1}$

Then $(f_1 \bar{g}_1)\cdot(g_1 \cdot f_2 \bar{g}_2)\cdot \cdots \cdot (g_{m-1} f_m) \simeq f$

and $g_i f_{i+1} \bar{g}_{i+1} : I \to A_i$

and $g_i f_{i+1} \bar{g}_{i+1}(0) = g_i(0) = x_0$

$g_i f_{i+1} \bar{g}_{i+1}(1) = \bar{g}_1(1) = x_0$

and $g_i(1) = f(s_i) = f_{i+1}(0)$, $f_{i+1}(1) = f(s_{i+1}) = \bar{g}_{i+1}(0)$

$\forall \alpha$ $A_\alpha \hookrightarrow X \rightsquigarrow \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$

$$\bigoplus_\alpha \pi_1(A_\alpha, x_0) \xrightarrow{\pi} \pi_1(X, x_0) \quad \text{product}$$

Lemma $\Rightarrow \pi$ is surjective.

Thm

$\pi_1(S^n) = 0$ for $n \geq 2$

Pf

$S^n = A_1 \cup A_2$, $A_i \cong \mathbb{R}^n$, $A_1 \cap A_2 \cong S^{n-1} \times \mathbb{R}$ (path-connected)

$\rightsquigarrow \pi_1(A_i, x_0) = 0$ $i = 1, 2$

$\Rightarrow \pi_1(A_1, x_0) \oplus \pi_1(A_2, x_0) = 0 \xrightarrow[\text{surjective}]{} \pi_1(S^n, x_0)$

Notice
$X \simeq Y \Rightarrow \pi_1(X) \simeq \pi_1(Y)$
with same base-pt

<u>Cor.</u> $\mathbb{R}^2 \not\cong \mathbb{R}^n$ for $n \neq 2$

<u>Pf</u> We show that $\mathbb{R}^2 - \{pt\} \not\simeq \mathbb{R}^n - \{pt\}$

We have $\mathbb{R}^n - \{pt\} \cong S^{n-1} \times \mathbb{R}$ (Let $pt = 0$.)

by $(x_1, \dots, x_n) \longmapsto \left(\frac{(x_1, \dots, x_n)}{\|(x_1, \dots, x_n)\|}, \log(\|(x_1, \dots, x_n)\|) \right)$

$y \cdot e^t \longleftarrow (\bar{y}, t)$, $\|\bar{y}\| = 1$

$$\pi_1(S^{n-1} \times \mathbb{R}) \simeq \pi_1(S^{n-1}) \times \pi_1(\mathbb{R}) \simeq \pi_1(S^{n-1}) = \begin{cases} \mathbb{Z} & n = 2 \\ 0 & n \neq 2 \end{cases}$$

<u>Prop</u> 1) $r : X \underset{\xleftarrow{i}}{\longrightarrow} A$, $ri = \mathbb{1}$ $\Rightarrow$ i_* is injective

2) A is a deformation retract $\Rightarrow$ i_* is isomorphism

<u>Proof</u> 1) $ri = \mathbb{1} \Rightarrow r_* i_* = \mathbb{1}$ $\Rightarrow$ i_* is injective

2) $r_t : X \to X$, deformation retract ; $r_0 = \mathbb{1}$, $r_t|_A = \mathbb{1}$, $r_1(X) \subseteq A$

then $\forall f : I \to X$, loop, $r_t f$ homotopy, $r_0 f = f$, $r_1 f(I) \subseteq A$

$\Rightarrow$ $i_*[r_1 f] = [i r_1 f] = [r_1 f] = [f]$; surjective

Notice

$H \underset{\xleftarrow{\rho}}{\hookrightarrow} G$ $\rightsquigarrow$ $\ker \rho \hookrightarrow G \twoheadrightarrow \underset{\text{subgp.}}{H}$.

$\Rightarrow$ G is an extension of H by $\ker \rho$.

Basepoint preserving homotopy

$\varphi_t : (X, x_0) \longrightarrow (Y, y_0)$, $\varphi_t(x_0) = y_0$ $\forall t \in I$

$\Rightarrow$ $\varphi_{0*}[f] = [\varphi_0 f] = [\varphi_1 f] = \varphi_{1*}[f]$.

Homotopy equivalence for spaces with basepoint.

$(X, x_0) \simeq (Y, y_0)$: $\varphi : (X, x_0) \longrightarrow (Y, y_0)$, $\psi : (Y, y_0) \longrightarrow (X, x_0)$

s.t. $\varphi\psi \simeq \mathbb{1}_Y$, $\psi\varphi \simeq \mathbb{1}_X$ fixing basepoints

Prop

$$\underset{\substack{\text{homotopy}\\ \text{equivalence}\\ \text{(drop basepoints)}}}{\varphi : X \longrightarrow Y} \quad \Rightarrow \varphi_* : \pi_1(X, x_0) \xrightarrow{\cong} \pi_1(Y, \overset{\varphi(x_0)}{\overset{\|}{y_0}})$$

iso for all $x_0 \in X$

Pf.

$$\pi_1(X, x_0) \xrightarrow{\varphi_*} \pi_1(Y, \varphi(x_0)) \xrightarrow{\psi_*} \pi_1(X, \psi\varphi(x_0)) \xrightarrow{\varphi_*} \pi_1(Y, \varphi(\psi(\varphi(x_0))))$$

Lemma:

$$\underset{\text{homotopy}}{\varphi_t : X \to Y}, \quad h : I \to Y; \ h(t) = \varphi_t(x_0)$$

$$\Rightarrow \quad \varphi_{0*} = \beta_h \varphi_{1*}$$

where $\beta_h[f] = [h \cdot f \cdot \bar{h}]$.

(Remember that β_h is an isomorphism)

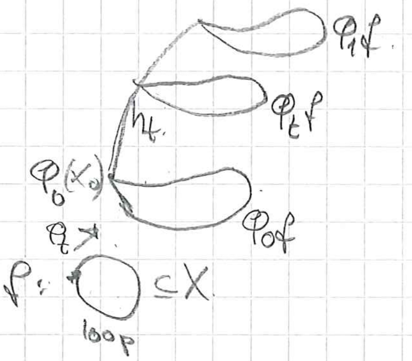
Proof

$$h_t = h|_{[0,t]} \ ; \ h_t(s) = h(ts) \ : \ h_t : I \to X.$$

If $f : I \to X$, $f(0) = f(1) = x_0$;

then $h_t \cdot (\varphi_t f) \cdot \bar{h}_t : I \longrightarrow X$ homotopy of loops

$$\left.\begin{array}{l} \text{where } h_t \cdot (\varphi_t f) \cdot \bar{h}_t(1) = h(0) = x_0, \\ h_t \cdot (\varphi_t f)\, \bar{h}_t(0) = h(0) = x_0 \end{array}\right\} \text{loop.}$$



$$\Rightarrow \varphi_{0*}[f] = [\varphi_0 f] = [h_t \cdot \varphi_t f \cdot \bar{h}] = \beta_h[\varphi_1 f] = \beta_h \varphi_{1*}[f].$$

Back to proof:

$$\underset{\substack{g_t\\ \text{homotopy}}}{(\psi\varphi)_* \simeq \mathbb{1}}, \quad \psi\varphi, \mathbb{1} : X \longrightarrow X$$

$$x_0 \longmapsto \psi\varphi(x_0)$$
$$x_0 \longmapsto \mathbb{1}(x_0) = x_0$$

$$h : I \to X, \ h(0) = x_0, \ h(1) = \psi\varphi(x_0)$$

$$\rightsquigarrow \beta_h[f] = [h \cdot f \cdot \bar{h}].$$

$$\Rightarrow (\psi\varphi)_* = \beta_h \ \text{iso} \Rightarrow \varphi_* \text{ is injective}$$

Same argument $(\varphi\psi)_* = \beta_{h'}$ iso $\Rightarrow \psi_*$ is injective

Composition of two injective maps is an isomorphism.

Pick $[f] \in \pi_1(Y, \varphi(x_0))$. Then $\psi_*[f] \in \pi_1(X, \psi\varphi(x_0))$

By iso-property $\exists [g] \in \pi_1(X, x_0)$ s.t. $\psi_*\varphi_*[g] = \psi_*[f]$

Injectivity of ψ_* gives $\varphi_*[g] = [f]$. Thus φ_* is surjective