1. $f_0 \cdot g_0 \underset{\varphi_t}{\simeq} f_1 \cdot g_1 \qquad g_0 \underset{\psi_t}{\simeq} g_1$

Define homotopy.

$$h_t(s) = \begin{cases} f_0\left(\frac{3s}{3-2t}\right) & 0 \le s \le 1 - \frac{2}{3}t \\ g_0(3s-3+2t) & 1-\frac{2}{3}t \le s \le 1 - \frac{1}{3}t \\ g_0(1-(3s-2)) & 1 - \frac{1}{3}t \le s \le 1 \end{cases}$$

$$\Rightarrow \quad h_0(s) = f_0(s), \quad h_1(s) = \begin{cases} f_0(3s) & [0, 1-\frac{2}{3}t] \\ g_0(3s-1) & [1-\frac{2}{3}t, 1-\frac{1}{3}t] \\ g_0(1-(3s-2)) & [1-\frac{1}{3}t, 1] \end{cases}$$

$$\Rightarrow h_1 = f_0 \cdot g_0 \cdot \bar{g}_0$$

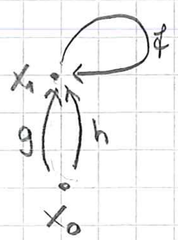
We can do the same for f_1 and g_1

$$\Rightarrow [f_0] = [f_0 \cdot g_0 \cdot \bar{g}_0] \text{ and } [f_1] = [f_1 \cdot g_1 \cdot \bar{g}_1]$$

But $\varphi_t \cdot \bar{\psi}_t$ is a homotopy from $f_0 \cdot g_0 \cdot \bar{g}_0$ and $f_1 \cdot g_1 \cdot \bar{g}_1$

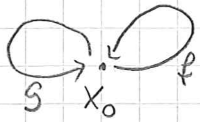
$$\Rightarrow [f_0] = [f_0 \cdot g_0 \cdot \bar{g}_0] = [f_1 \cdot g_1 \cdot \bar{g}_1] = [f_1].$$

2. Let g, h be two homotopic paths from x_0 til x_1, i.e. $[g \cdot \bar{h}] = 0 \in \pi_1(X, x_0)$. Let $[f] \in \pi_1(X, x_1)$



$$\Rightarrow \beta_h[f] = [h \cdot f \cdot \bar{h}] = [g \cdot \bar{h} \cdot h \cdot f \cdot \bar{h} \cdot h \cdot \bar{g}] = [g \cdot f \cdot \bar{g}] = \beta_g[f].$$

3.



$1\!\!1, g$: paths from x_0 to x_0. By assumption

$$\beta_{1\!\!1}[f] = \beta_g[f]$$

$$\Rightarrow [f] = [g f \bar{g}] \Rightarrow [f] \cdot [\bar{f}] = [g f \bar{g}][\bar{f}]$$

$$\Rightarrow 0 = [g f \bar{g} \bar{f}] \in \pi_1(X, x_0)$$

$$\Rightarrow [g][f][g]^{-1}[f]^{-1} = e.$$

5. (a) $S^1 \to X$ is null homotopic i.e. $[S^1 \xrightarrow{\gamma} X] = 0 \in \pi_1(X, x_0)$ $\forall x_0 \in \operatorname{im}\gamma$.

(b) $S^1 \to X$ extends to $D^2 \to X$

(c) $\pi_1(X, x_0) = 0 \quad \forall x_0 \in X$

(c) $\Rightarrow$ (a) ok.

(b) $\Rightarrow$ (c) $\gamma : S^1 \to X$, loop, etends to $f : D^2 \to X$, i.e. $f|_{\partial D^2} = \gamma$

Let $h_t(x) = (1-t)x + t\,x_0$ for all $x \in D^2$ and some fixed $x_0 \in S^1$

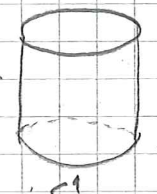
$\Rightarrow \quad h_t : D^2 \to D^2, \; h_0 = \mathbb{1}_{D^2}, \; h_1 = x_0 \in S^1$

$\rightsquigarrow \quad f h_t : D^2 \to X$ and $\gamma_t = f h_t|_{\partial D^2} \to X$

Then $\gamma_0 = f h_0|_{S^1} = f|_{S^1} = \gamma$ and $\gamma_1 = f h_1|_{S^1} = f(x_0) \stackrel{!}{=} 0$.

$\Rightarrow [\gamma_0] = [0]$

(a) $\Rightarrow$ (b)



I

S^1

$\xrightarrow{x_0} X$

$\xrightarrow{h} X$ homotopy.

$\xrightarrow{f} X$

h is the restriction of the kontinuous map where $h = x_0$ on $D^2 \times \{1\}$

The pair (D^2, S^1) has the homotopy extension property $\Rightarrow \exists\, \tilde{h} : D^2 \times I \to X$

$\rightsquigarrow \tilde{h}|_{D^2 \times \{0\}} : D^2 \to X$.

6. $[S^1, X] = \{ [f : S^1 \to X] \}$.

Natural map $\Phi : \pi_1(X, x_0) \to [S^1, X]$ (forgetting base pt) Let $\gamma : S^1 \to X$

1) X path-connected $\Rightarrow$

$h : I \to X$

$1 \mapsto x_0$

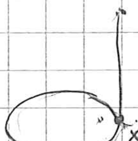
$0 \mapsto x_1 \in S^1$

path

$$\begin{tikzcd} S^1 \times I \arrow[r, dashed, "\exists \text{ homotopy}"] & X \\ S^1 \times \{0\} \cup \{x_1\} \times I \arrow[u, hook] \arrow[ur, "\gamma \cup h"'] & \end{tikzcd}$$

(closed)

$(S^1, \{s_0\})$ has the homotopy extension property



x_0

x_1

$H(x,1) : S^1 \to X$ loop $\qquad H(s_0, 1) = x_0$.

9. $A_1, A_2, A_3 \subset \mathbb{R}^3$ compact : $\exists!\ P \subset \mathbb{R}^3$ plane such that P divides each A_i into two pieces of equal measure

Pf

$$\text{Plane} \subset \mathbb{R}^3 \xrightarrow{N} \text{Normal vector } \frac{1}{\|\ \|} \in S^2.$$

Let $P_i \in N^{-1}(s)$, $s \in S^2$ divide A_i in two pieces of equal measure

$\vec{d}_s(x,y) = -\vec{d}_{\bar{s}}(y,x)$ Let d_j = directed distance between P_1 and P_j along s.

i.e. $\vec{d}_s(\vec{s} \cap P_1, \vec{s} \cap P_j)$. Then $D = (d_1, d_2) : S^2 \to \mathbb{R}^2$ is a continuous map.

By the Borsuk-Ulam Thm $\exists s \in S^2$ s.t. $D(s) = D(-s)$.

But by definition $D(-s) = -D(s)$ $\Rightarrow$ $D(s) = 0$.

$\Rightarrow$ $P_1 = P_2 = P_3$ (parallel through a common pt)

12.

$\gamma : \pi_1(S^1) \to \pi_1(S^1)$ · generator for $\pi_1(S^1)$: $[\omega]$ where $\omega(s) = e^{2\pi i s}$ $0 \le s \le 1$

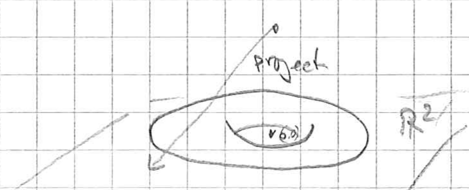
$\rightsquigarrow$ $\gamma([\omega]) = [\omega]^n$ for some $n \in \mathbb{Z}$.

But $[\omega]^n = [\omega_n]$ where $\omega_n(s) = e^{2\pi i n s}$

Now let $\varphi : S^1 \to S^1$ be given by $\varphi(z) = z^n$

$\Rightarrow$ $\gamma([\omega]) = [\omega]^n = [\omega_n] = [\varphi\, \omega] = \varphi_*([\omega])$

8.



$f(\underline{x}) = f(\underline{y})$, but $\underline{x} \neq -\underline{y}$.

14. $\pi_1(X \times Y) \longrightarrow \pi_1(X) \times \pi_1(Y)$

$[f] \in \pi_1(X \times Y)$ loop based at (x_0, y_0) $\quad p_1 f : I \to X$ continous $\quad p_1 f(0) = p_1(x_0, y_0) = x_0$

$p_1 f(1) = p_1(x_0, y_0) = x_0$

$$\begin{array}{ccc} & X \times Y & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ X & & Y \end{array}$$

$\Rightarrow [f] \mapsto [p_1 f]$

$= p_{1*}[f]$.

15.
$$I \xrightarrow{h} X \xrightarrow{f} Y$$
$$0 \longmapsto x_0$$
$$1 \longmapsto x_1$$

$$\begin{tikzcd} \pi_1(X, x_1) \arrow[r, "\beta_h"] \arrow[d, "f_*"] \arrow[dr, phantom, "\circ"] & \pi_1(X, x_0) \arrow[d, "f_*"] \\ \pi_1(Y, f(x_1)) \arrow[r, "\beta_{fh}"'] & \pi_1(Y, f(x_0)) \end{tikzcd}$$

Pf.
$$f_* \beta_h[\gamma] = f_*[h\cdot\gamma\cdot\bar{h}] = [f\circ(h\cdot\gamma\cdot\bar{h})]$$
$\gamma : I \to X$ $\quad = [fh \cdot f\gamma \cdot f\bar{h}]$

$\gamma(0)=\gamma(1)=x_1.$ $\quad = [fh \cdot f\gamma \cdot \overline{fh}] = \beta_{fh}[f\gamma] = \beta_{fh} f_*[\gamma]$

1) $f\circ(g\cdot h) = fg\cdot fh.$

$$\Rightarrow f_*\beta_h = \beta_{fh} f_*$$

2) $f\bar{h} = \overline{fh}$

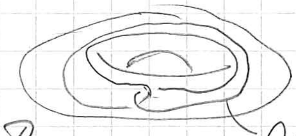
16. a) $\mathbb{R}^3 \underset{\iota}{\overset{r}{\rightleftarrows}} S^1$ $\quad \{e\} = \pi_1(\mathbb{R}^3) \underset{\iota_*}{\overset{r_*}{\rightleftarrows}} \pi_1(S^1) = \mathbb{Z}$

$r_*\iota_* = \mathbb{1}_{\mathbb{Z}}$ impossible

b) $S^1\times S^1 \underset{\iota}{\overset{r}{\rightleftarrows}} S^1\times D^2$

$\pi_1 = \mathbb{Z}\times\mathbb{Z}$ $\quad \pi_1 = \mathbb{Z}.$ $\quad$ Not possible: $\mathbb{Z}\times\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}$, $\mathbb{1}$

c) $X = S^1\times D^2$



$\pi_1(S^1\times D^2) \cong \mathbb{Z}$ $\quad A \cong S^1$ nullhomotopic in X

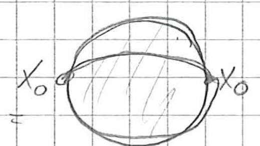
$$A \hookrightarrow S^1\times D^2 \longrightarrow A$$
$$\mathbb{Z} \hookrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}$$
$$\langle\omega\rangle \longmapsto 0$$

nullhomotopic in X, even if non-zero in $\pi_1(A, x_0)$.

d) $X = D^2 \vee D^2$ $\quad \pi_1(X, x_0) = \pi_1(D^2, x_0) * \pi_1(D^2, x_0) = 0.$

$x_0 \in D^2 \cap D^2$

$\pi_1(S^1\vee S^1) = \langle a\rangle * \langle b\rangle$ $\quad$ impossible

e) $X =$ 

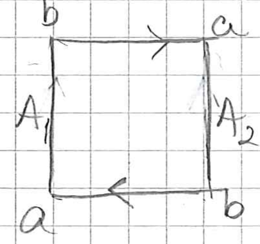
$\cup|$

$A = S^1 \vee S^1$

$\pi_1(X_{x_0}) \simeq \mathbb{Z}$ $\qquad \pi_1(A, x_0) = \langle a \rangle * \langle b \rangle$

$\neq 1\!\!1.$

$$\langle a \rangle * \langle b \rangle \longrightarrow \mathbb{Z} = \langle a \rangle \longrightarrow \langle a \rangle * \langle b \rangle$$

f) 

$A = A_1 \cup A_2$

$\pi_1(X) \simeq \mathbb{Z}$, $\pi_1(A) = \mathbb{Z}$

$$\pi_1(A) \longrightarrow \pi_1(X) \xrightarrow{\ r_*\ } \pi_1(A)$$

$[\omega] \longmapsto [\omega]^2$

generator

$r_*[\omega] = [\omega]^m$

But $[\omega] \longmapsto [\omega]^{2m} = [\omega]^1$

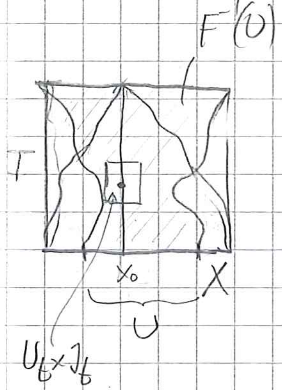
$\Rightarrow 2m = 1$, $m \in \mathbb{Z}$ impossible

5. (again).

$F : X \times I \longrightarrow X$

deformation retract $\Rightarrow F(x,0) = 1\!\!1_X$, $F(x,1) = \{x_0\}$

and $F(x_0, t) = x_0 \quad \forall t \in I$



$x_0 \in U \subset X$; $F^{-1}(U) = \{(x,t) \in X \times I \,;\, F(x,t) = f_t(x) \in U\}$
nbh.

We have $\{x_0\} \times I \subseteq F^{-1}(U)$ by $F(x_0, t) = x_0 \ \forall t$

Pick $t \in I$. Then $\exists\ x_0 \in \underset{\text{nbh.}}{U_t} \subset X$ and $t \in J_t \subset I$

such that $U_t \times J_t \subseteq F^{-1}(U)$

By compactness ; $\{x_0\} \times I \subseteq \bigcup_{\text{finite}} (U_t \times J_t)$ Put $V = \bigcap_{\text{finite}} U_t \ni x_0$ open

$\Rightarrow V \times I \subseteq \bigcup_{\text{finite}} (U_t \times J_t) \subseteq F^{-1}(U)$

$\Rightarrow f_t(V) \subset U \quad \forall t \in I$ defines homotopy