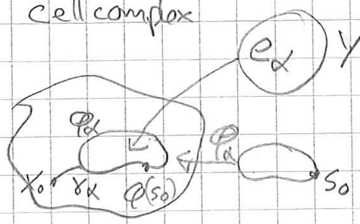
Setup: cell complex X

$$X \xhookrightarrow{i} Y = X \sqcup_{\varphi_\alpha} (\cup e_\alpha)$$



$(\varphi_\alpha : \operatorname{im}\varphi_\alpha)$

$\gamma_\alpha \varphi_\alpha \bar{\gamma}_\alpha$ loop at x_0

possible $\not\simeq x_0$, but after attaching $i \simeq x_0$

$\rightsquigarrow \quad N \subseteq \pi_1(X, x_0)$

$$\langle \gamma_\alpha \varphi_\alpha \bar{\gamma}_\alpha \mid \alpha \rangle \subseteq \ker \pi_1(X, x_0) \xrightarrow[i_*]{} \pi_1(Y, x_0)$$

normal subgroup

Proposition:

a) $\pi_1(Y) \simeq \pi_1(X)/N$

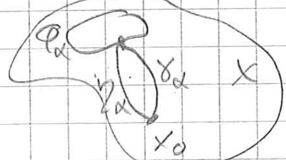
b) Attaching n-cells for $n > 2$

$\Rightarrow \pi_1(X, x_0) \simeq \pi_1(Y, x_0)$

c) $X^2 \xhookrightarrow{i} X$ 2-skeleton

$\rightsquigarrow \pi_1(X^2, x_0) \simeq \pi_1(X, x_0)$

Observation: N is independent of γ_α.

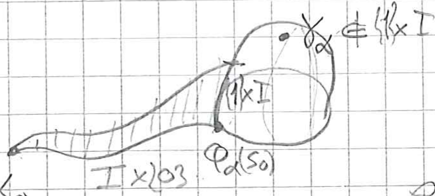


$$\eta_\alpha \varphi_\alpha \bar{\eta}_\alpha = \eta_\alpha (\bar{\gamma}_\alpha \gamma_\alpha) \varphi_\alpha (\bar{\gamma}_\alpha \gamma_\alpha) \bar{\eta}_\alpha$$
$$= (\eta_\alpha \bar{\gamma}_\alpha)(\gamma_\alpha \varphi_\alpha \bar{\gamma}_\alpha)\overline{(\eta_\alpha \bar{\gamma}_\alpha)}$$

Conjugate element in normal subgroup.

Proof a)

$Y \rightsquigarrow Z$



$$Y \underset{i}{\overset{r}{\rightleftarrows}} Z$$

$\pi_1(Y) \simeq \pi_1(Z)$ retracts.

$A = Z \setminus \bigcup_\alpha \{y_\alpha\} \quad B = Z - X$

$e_\alpha \setminus \{y_\alpha\}$ deformation retracts to $\varphi_\alpha \Rightarrow A$ deformation retracts onto X

B is contractible

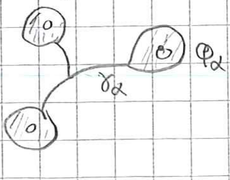
Van Kampen applied to $\{A, B\}$:

$$\begin{tikzcd} & \pi_1(A) \arrow[dr] & \\ \pi_1(A \cap B) \arrow[ur, "j"] \arrow[dr] & & \pi_1(A \cup B) \\ & \pi_1(B) = \{0\} \arrow[ur] & \end{tikzcd}$$

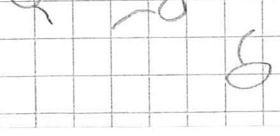
$\Rightarrow \pi_1(A \cup B) \simeq \pi_1(A)/\operatorname{im} j$

But $\pi_1(A) \simeq \pi_1(X)$, $\pi_1(A \cup B) \simeq \pi_1(Z) \simeq \pi_1(Y)$

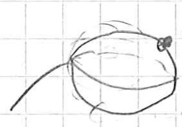
and $\pi_1(A \cap B)$:



Apply Van Kampen to



Consider $j(\pi_1(A \cap B))$.

b) Same as a), but  remove this point

$\mathbb{R}^n \setminus \{0\} \simeq S^{n-1}$ contractible

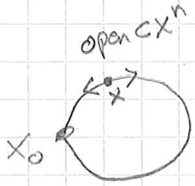
c) $X^2 \hookrightarrow X^3 \hookrightarrow X^4 \hookrightarrow \cdots \hookrightarrow X$ finite dimensional.

Infinite-dimensional:

$f: I \to X$ (loop), basept $x_0 \in X^2$

compact image $\Rightarrow f(I) \subset X^n$ for some n.

$\Rightarrow \pi_1(X^2, x_0) \to \pi_1(X, x_0)$ surjective


open $\subset X^n$

x_0

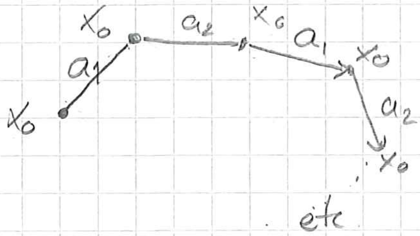
Compact $\Rightarrow$ finite cover.

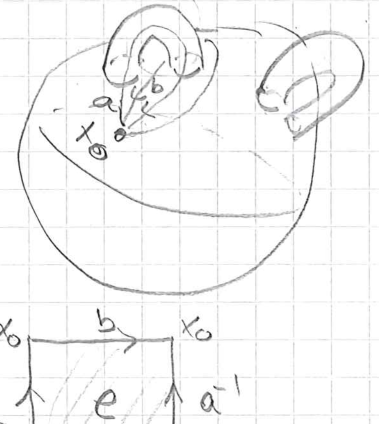
Suppose $f: I \to X$ is null homotopic in X by homotopy $F: I \times I \to X$. Compact $\Rightarrow F(I \times I) \subset X^n$ for some n.

But $\pi_1(X^2, x_0) \hookrightarrow \pi_1(X^n, x_0)$

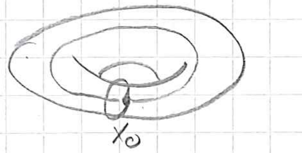
$\Rightarrow f$ 0-homotopic also in X^2.

Application:

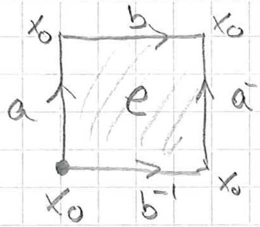




At torus



or

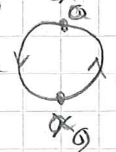


$$\Rightarrow \pi_1(M_g) = \langle a_1, b_1, a_2 b_2, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle.$$

Notice $\pi_1(M_g) / [\pi_1(M_g), \pi_1(M_g)] \simeq \mathbb{Z}^{2g}$

$$\Rightarrow M_g \not\cong M_h \quad g \neq h.$$

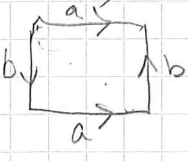
Non-orientable surfaces

N_1: 

$\pi_1(N_1) = \langle a \mid a^2 \rangle \simeq \mathbb{Z}_2$

$aba^{-1}b \mapsto aaca^{-1}ac = a^2c^2$

$N_g \not\cong N_h$.

N_2: 

$\pi_1(N_2) = \langle a, b \mid aba^{-1}b \rangle \simeq \langle a, c \mid a^2c^2 \rangle$

$\pi_1(N_2)/_{ab} \simeq \overset{b}{\mathbb{Z}_2} \times \overset{a}{\mathbb{Z}}$

$a \mapsto a$

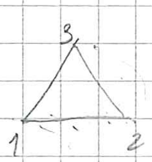
$b \mapsto ac$, $a^{-1}b \mapsto a^{-1}ac = c$

Cor. G group. $\Rightarrow \exists X_G$ 2-dim cell complex s.t. $\pi_1(X_G) \cong G$.

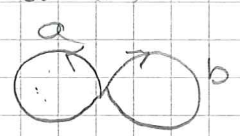
Pf. $G = \langle g_\alpha \mid r_\beta \rangle$ general relations

X_G : $\bigvee_\alpha S^1_\alpha$, attach e_β according to the rule r_β

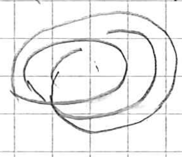
Ex. $G \simeq S_3 = \langle a, b \mid a^2, b^3, abab \rangle = \{e, \underset{(12)}{a}, \underset{(123)}{b}, \underset{(132)}{b^2}, \underset{(23)}{ab}, \underset{(13)}{ba}\}$



$a = (12)$ $b = (123)$ $abab = (12)(123)(12)(123) = id$



$G \simeq \mathbb{Z}_3 = \langle a \mid a^3 \rangle$.

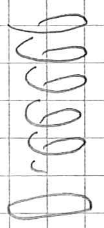


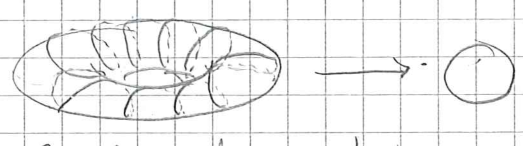
Attach a 2-cell

Covering space

$$\begin{array}{ccccc} \tilde{X} & \supseteq & p^{-1}(U) = \bigsqcup_i U_i, & U_i \xrightarrow[p]{\cong} U \\ p\downarrow & & \downarrow & \text{homeomorphic} \\ X & \supseteq & U \ni x & (U \text{ is evenly covered}) \\ & & \text{open} & \end{array}$$

Covers of S^1





curve winding n times round a torus.

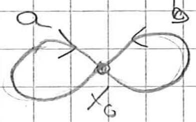
$$S^1 \times S^1 \xrightarrow{\quad p \quad} S^1$$

$$t \in [0,1] \qquad (t \cdot 2\pi, nt \cdot 2\pi) \longmapsto nt \cdot 2\pi \pmod{2\pi}$$

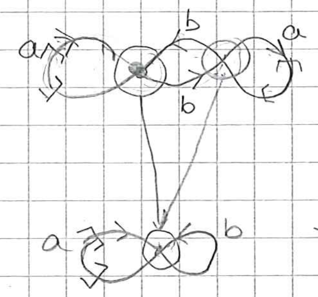
$$nt = r \pmod 1 \qquad = p^{-1}(r \cdot 2\pi)$$

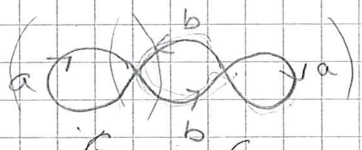
$$0 \leq r \leq 1$$

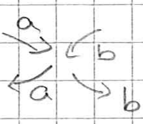
Covers of $S^1 \vee S^1$:

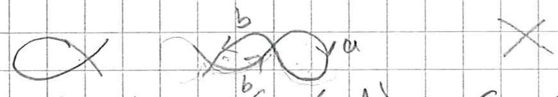


$$\pi_1 = \mathbb{Z} * \mathbb{Z} = \langle a, b \rangle.$$







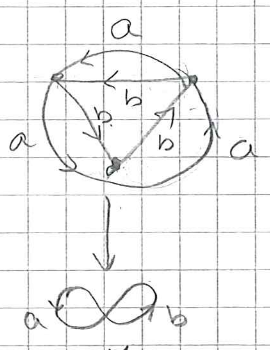


$G_1 = \langle a \rangle$ $\qquad G_2 = \langle c, d \rangle$ $\qquad G_1 \cap G_2$

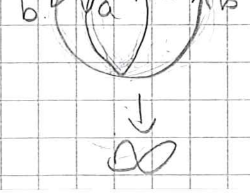
$$\Rightarrow \qquad \pi_1 = \langle a, c, d \rangle \qquad c = b^2$$

$$bab^{-1} = d$$

$$= \langle a, b^2, bab^{-1} \rangle$$



$$\pi_1(\tilde{X}) = \langle ab^{-1}, b^3, a^3, b^{-1}a \rangle$$



$$\pi_1(\tilde{X}) = \langle a^2, b^2, ba^{-1}, ba, ab, b^{-1}a, a^{-1}b, ab^{-1} \rangle$$

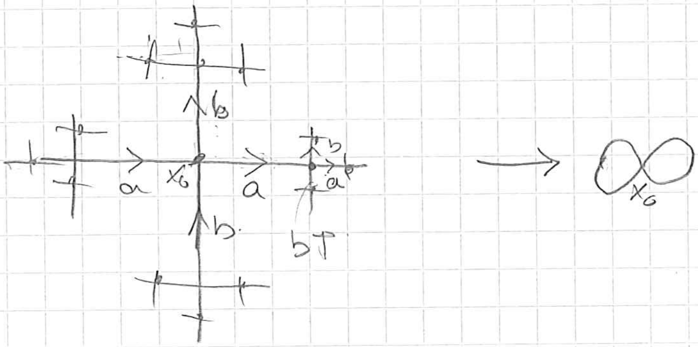
$$= \langle a^2, b^2, ab \rangle$$

$$b^{-1}a^{-1} \cdot a^2 = b^{-1}a \qquad a^{-2}ab = a^{-1}b$$

$$b^2 \cdot b^{-1}a^{-1} = ba^{-1} \qquad abb^{-2} = ab^{-1}$$

$$b^2 b^{-1} a^{-1} a^2 = ba$$

Simply-connected covering space



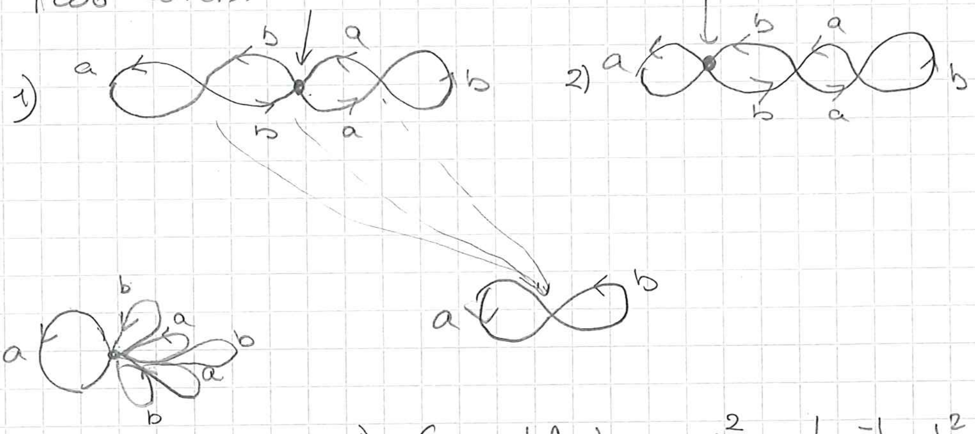
Universal cover

continue....

1) $\pi_1(\tilde{X}) = 0$.

2) Cover of every other cover

Two covers. basept: $\tilde{x}_0$



$3 \times b$
$3 \times a$.

1) Generated by $a^2, aba^{-1}, b^2, bab^{-1} = G_1$

2) Generated by $a, b^2, ba^2b^{-1}, baba^{-1}b^{-1} = G_2$

Conjugate G_2 $b^{-1}ab,\ b^{-1}b^2b,\ b^{-1}ba^2b^{-1}b,\ b^{-1}baba^{-1}b^{-1}b$

$\quad\quad\quad\quad\quad\ \ = b^{-2}bab^{-1}b^2,\ = b^2,\ = a^2,\ = aba^{-1}$

$\Rightarrow$ $G_1 \approx G_2$ by conjugation

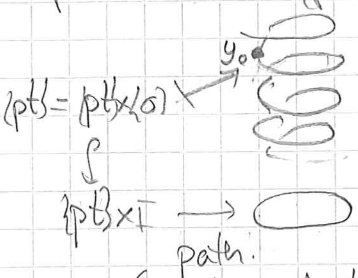
Remind: Homotopy lifting property

$$\begin{tikzcd} Y\times\{0\} \arrow[r, "\tilde{f}_0"] \arrow[d, hook] & \tilde{X} \arrow[d, "P"] \\ Y\times I \arrow[r, "F"'] \arrow[ur, dashed, "\exists!\tilde{F}"] & X \end{tikzcd}$$

homotopy

$\tilde{F}(y,t) = \tilde{f}_t(y)$.

Special case $Y = \{pt\}$



path
(homotopy of pts)