$x_0 \in X$, $\gamma : I \to X$ path

$[\gamma]$ homotopy class of paths with fixed end points $\gamma(0)$ and $\gamma(1)$

Define:

$$\tilde{X} = \{ [\gamma] \mid \gamma : I \to X,\ \gamma(0) = x_0 \}$$

$$p : \tilde{X} \to X$$

$$[\gamma] \longmapsto \gamma(1) \quad \text{well-defined}$$

1) X path-connected $\Rightarrow$ p is surjective $\Rightarrow$: $\forall x_1 \in X$, $\exists \gamma : I \to X$ such that $\gamma(1) = x_1$, $\gamma(0) = x_0$.

2) $\tilde{X}$ as a topological space: $U \subseteq X$ open, define $U_{[\gamma]} = \{ [\gamma \cdot \eta] \mid \eta : I \to U \subseteq X,\ \eta(0) = \gamma(1) \}$

$\mathcal{U} = \underline{\{ U \mid \text{path-conn}, \text{semiloc. s-c} \}}$

U path-connected $\Rightarrow p : U_{[\gamma]} \to U$ surjective

Injective? $\eta_1, \eta_2 : I \to U$, $\eta_1(1) = \eta_2(1)$

Then $[\eta_1 \cdot \bar{\eta}_2] \in \pi_1(U)$. $\iota : U \to X$

X semilocally simply-connected

$\overset{\text{def}}{\Rightarrow}$ $\forall x \in X$, $\exists\, x \in U \subset X$ open s.t. $\pi_1(U) \to \pi_1(X)$ trivial

Semilocally simply-connected $\Rightarrow$

$\iota_* \pi_1(U) = 0$ i.e. $[\eta_1 \cdot \bar{\eta}_2] = 0 \in \pi_1(X)$

$\Rightarrow [\eta_1] = [\eta_2]$

Notice alson that $U_{[\gamma]} = U_{[\gamma']}$ if $[\gamma'] \in U_{[\gamma]}$.

Pf

$\gamma' = \gamma \cdot \eta \Rightarrow x \in U_{[\gamma']}$; $x = [\gamma \cdot \eta \cdot \mu] \in U_{[\gamma]}$

$y \in U_{[\gamma]}$; $y = [\gamma \cdot \mu] = [\gamma \cdot \eta \cdot \bar{\eta} \cdot \mu] = [\gamma' \cdot \bar{\eta} \cdot \mu] \in U_{[\gamma']}$

3) $\{ U_{[\gamma]} \}$ form a topology of $\tilde{X}$:

$U_{[\gamma]}$, $V_{[\gamma']}$

$[\gamma''] \in U_{[\gamma]} \cap V_{[\gamma']} \Rightarrow U_{[\gamma]} = U_{[\gamma'']}$ and $V_{[\gamma']} = V_{[\gamma'']}$.

If $W \in \mathcal{U}$, $W \subset U \cap V$, $\gamma''(1) \in W \Rightarrow W_{[\gamma'']} \subset U_{[\gamma'']} \cap V_{[\gamma'']}$ and $[\gamma''] \in W_{[\gamma'']}$.

4) p homeomorphic / continuous

$$\{V_{[\gamma']} \subset U_{[\gamma]}\} \xrightarrow{p} \{U \subset V \in \mathcal{U}\}$$

$$p V_{[\gamma']} = V$$

$$p^{-1}(V) \cap U_{[\gamma]} = V_{[\gamma']} \quad \forall [\gamma'] \in U_{[\gamma]} \text{ with } \gamma'(1) \in V$$

since $V_{[\gamma']} \subset U_{[\gamma']} = U_{[\gamma]}$

and $p V_{[\gamma']} = V$.

5) Covering

$U \in \mathcal{U}$ $\quad p^{-1}(U) = \bigsqcup_{[\gamma]} U_{[\gamma]}$.

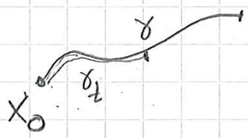
because $[\gamma''] \in U_{[\gamma]} \cap U_{[\gamma']}$, then $U_{[\gamma']} = U_{[\gamma'']} = U_{[\gamma]}$

6) $\tilde{X}$ simply-connected

Let $[\gamma] \in \tilde{X}$. Define homotopy $\mathcal{T}: I \times I \to X$

by $\mathcal{T}(s,t) = \gamma_t(s)$

and $\gamma_t(s) = \gamma(s)$, $s \in [0,t]$; $\gamma_t(s) = \gamma(t)$ $s \in [t,1]$

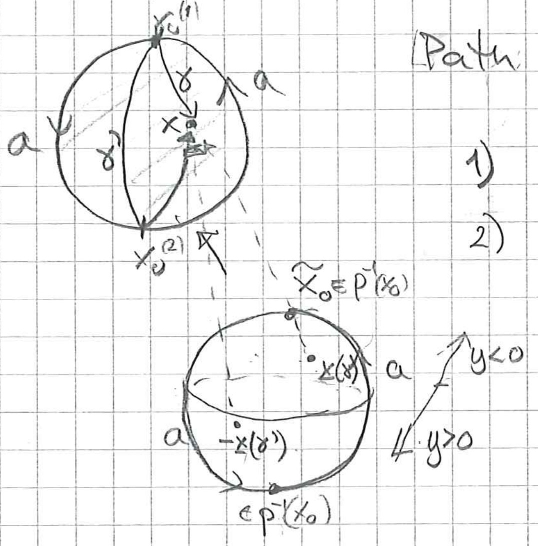


$\Rightarrow$ $t \mapsto [\gamma_t]$ path in $\tilde{X}$; $t=0$: $[x_0]$, $t=1$: $[\gamma]$
$I \longrightarrow \tilde{X}$

$\Rightarrow$ $\tilde{X}$ is path-connected

$[\tilde{\gamma}] \in \pi_1(\tilde{X}, [x_0])$, then $p_*([\tilde{\gamma}])$: loops in X, at x_0, that lifts to loop in $\tilde{X}$ at $[x_0]$. By the homotopy $\mathcal{T}$ these loops are trivial in $\pi_1[X, x_0]$. Since p_* is injective, $[\tilde{\gamma}] = 0$.

$X = \mathbb{RP}^2$



Path $\gamma : I \longrightarrow X$, $\gamma(0) = x_0$

1) $\#\{s \in I \mid \gamma(s) = x_0\}$ odd

2) $\#\{s \in I \mid \gamma(s) = x_0\}$ even

Paths from $x_0^{(1)}$: $y > 0$

Paths from $x_0^{(1)}$, via $x_0^{(2)}$: $y < 0$

Prop.

X path-connected, locally path-connected, semilocally simply-connected, $H \subset \pi_1(X, x_0)$ subgp $\Rightarrow \exists$ $p : X_H \to X$ covering s.t. $p_* \pi_1(X_H, \tilde{x}_0) = H$ for some $\tilde{x}_0 \in X_H$

Pf. Let $[\gamma], [\gamma'] \in \tilde{X}$ (simply-connected universal cover)

Put $[\gamma] \sim [\gamma']$ if $\gamma(1) = \gamma'(1)$ and $[\gamma \cdot \bar{\gamma}'] \in H$

$\sim$ is an equivalence relation:

R: $[\gamma] \sim [\gamma]$.

S: $[\gamma] \sim [\gamma'] \Rightarrow [\gamma'] \sim [\gamma]$

T: $[\gamma] \sim [\gamma'], [\gamma'] \sim [\gamma''] \Rightarrow \gamma(1) = \gamma'(1) = \gamma''(1)$
and $[\gamma \cdot \bar{\gamma}''] = [\gamma \cdot \bar{\gamma}' \cdot \gamma' \bar{\gamma}''] = [\gamma \bar{\gamma}'] \cdot [\gamma' \bar{\gamma}'']$

$$\rightsquigarrow \quad X_H = \tilde{X} \big/ [\gamma] \sim [\gamma'] \xrightarrow{[\gamma] \mapsto \gamma(1)} X$$

Notice: $\gamma(1) = \gamma'(1)$, then $[\gamma] \sim [\gamma']$ iff $[\gamma \cdot \eta] \sim [\gamma' \cdot \eta]$.

$\Rightarrow$ If two points in neighborhoods are identified then the two neighborhoods & $\Rightarrow$ Covering

For $\tilde{x}_0 \in X_H$ basept. ; $[\tilde{x}_0] \in \tilde{X}$ constant path $\Rightarrow$ $\gamma : I \to X$, $\gamma(0) = \gamma(1) = x_0$ loop

$\Rightarrow \tilde{\gamma} : I \to \tilde{X}$, $\tilde{\gamma}(0) = [\tilde{x}_0]$ lifting path. $\tilde{\gamma}(1) = [\gamma]$

$\Rightarrow$ $\tilde{X} \longrightarrow X_H$

$\gamma \longmapsto$ path in X_H : loop iff $[\gamma] \sim [\tilde{x}_0]$

or $[\gamma \cdot \tilde{x}_0] = [\gamma] \in H$.

QED.

Prop.

X
path-connected
loc. path connected

$$\begin{tikzcd}
\tilde{x}_1 \arrow[r, mapsto] \arrow[d, phantom, "\cap" rotate=90] & \tilde{x}_2 \arrow[d, phantom, "\cap" rotate=90] \\
\tilde{X}_1 \arrow[r, "f"] \arrow[dr, "p_1"'] & \tilde{X}_2 \arrow[d, "p_2"] \\
& X
\end{tikzcd}$$

$X \ni x_0 = p_1(\tilde{x}_1) = p_2(\tilde{x}_2)$

Isomorphic $\Longleftrightarrow$ $p_{1*}\pi_1(\tilde{X}_1, \tilde{x}_1) = p_{2*}\pi_1(\tilde{X}_2, \tilde{x}_2)$

Pf. $p_2 f = p_1$, $p_2 = p_1 f^{-1}$

$[p_1 \gamma] \in p_{1*}\pi_1(\tilde{X}_1, \tilde{x}_1)$, then $[p_1\gamma] = [p_2 f \gamma] \in p_{2*}\pi_1(\tilde{X}_2, \tilde{x}_2)$

and $[p_2 \eta] \in p_{2*}\pi_1(\tilde{X}_2, \tilde{x}_2)$, then $[p_2 \eta] = [p_1 f^{-1} \eta] \in p_{1*}\pi_1(\tilde{X}_1, \tilde{x}_1)$

$\Longrightarrow p_{1*}\pi_1(\tilde{X}_1, \tilde{x}_1) = p_{2*}\pi_1(\tilde{X}_2, \tilde{x}_2)$

Suppose $p_{1*}\pi_1(\tilde{X}_1, \tilde{x}_1) = p_{2*}\pi_1(\tilde{X}_2, \tilde{x}_2)$

By lifting criterion $\exists$ lifting of p_1; $\tilde{p}_1 : (\tilde{X}_1, \tilde{x}_1) \to (\tilde{X}_2, \tilde{x}_2)$

with $p_2 \tilde{p}_1 = p_1$.

$$\begin{tikzcd}
\{\tilde{x}_1\} \arrow[r, dashed] & \{\tilde{x}_2\} \subset \tilde{X}_2 \arrow[d, "p_2"] \\
\tilde{X}_1 \arrow[r, "p_1"'] \arrow[ur, dashed, "\tilde{p}_1"] & X
\end{tikzcd}$$

with $\{\tilde{x}_1\} \subset \tilde{X}_1$.

Similarly we find $\tilde{p}_2$.

By unique lifting property $\tilde{p}_1 \tilde{p}_2 = 1$, $\tilde{p}_2 \tilde{p}_1 = 1$.

since they fix basepoints.

QED.

<u>Rep.</u>

X path-connected, locally path-connected, semilocally simply-connected, $H \subseteq \pi_1(X, x_0)$ subgp. $\Rightarrow \exists$ $\begin{array}{c} X_H \\ \downarrow p \\ X \end{array}$ covering s.t. $p_* \pi_1(X_H, \tilde{x}_0) = H$ for some $\tilde{x}_0 \in X_H$

Pf. Put $[\gamma] \sim [\gamma'] \Leftrightarrow \gamma(1) = \gamma'(1)$ and $[\gamma \cdot \bar{\gamma}'] \in H$.

$[\gamma], [\gamma'] \in \tilde{X}$

$$X_H = \tilde{X}/_\sim \longrightarrow X \quad \text{covering}$$
$$[\gamma] \longmapsto \gamma(1)$$

For $[\gamma] \in \tilde{X}$, define $\gamma_t : I \longrightarrow X$, $\begin{cases} \text{path in } X \\ \text{pt in } \tilde{X} \end{cases}$
path or pt

$$\gamma_t(s) = \begin{cases} \gamma(s) & 0 \leq s \leq t \\ \gamma(t) & t \leq s \leq 1 \end{cases}$$

Notice: $\gamma_0(s) = \gamma(0)$ constant path

$\gamma_1(s) = \gamma(s) \Rightarrow [\gamma_1] = [\gamma]$.

$$\begin{tikzcd} \{\alpha\} \arrow[r, mapsto] \arrow[d, phantom, "\cap"] & \tilde{x}_0 \in \tilde{X} \arrow[d, "p"] \\ I \arrow[ur, "\exists! \tilde{\gamma}"] \arrow[r, "\gamma"] & X \end{tikzcd}$$

p (evaluating at 1)

Put $[x_0] = \tilde{x}_0$.
constant path

Notice that γ_t defines map $\tau : I \longrightarrow \tilde{X}$ by $\tau(t) = [\gamma_t]$ and

$$p\tau(t) = p[\gamma_t] = \gamma_t(1) = \gamma t$$
$$\Rightarrow p\tau = \gamma$$

By uniqueness of lifting $\tilde{\gamma} = \tau$

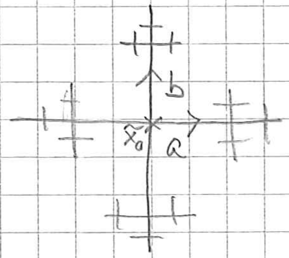
$$\Rightarrow \tilde{\gamma}(1) = \gamma_1 = \gamma$$

$\Rightarrow \tilde{\gamma}$ is a path from $[x_0]$ to $[\gamma]$.

$\tilde{\gamma}$ projects to path in X_H, loop if $[\gamma] \sim [x_0]$, i.e. $[\gamma] \in H$

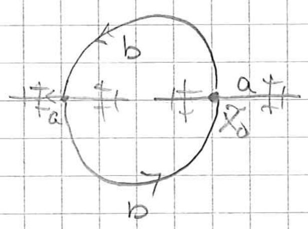
and vice versa.

1)

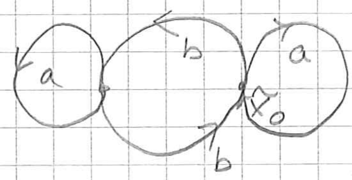


Universal cover of ∞.

$b^2 \in H$



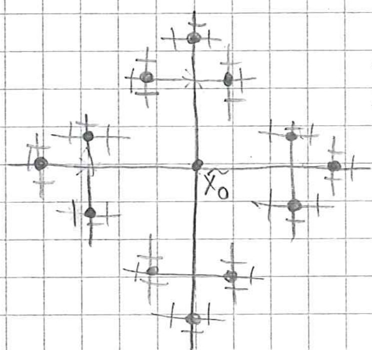
$a \in H$



Index $\langle a, b^2 \rangle$ in $\langle a, b \rangle$: 2.

$H = \langle a, b^2 \rangle$

2) $H = \langle ab, ba, a^2 \rangle$ $\quad gg'^{-1} = ab \Rightarrow g = abg'$



$\bullet : \tilde{x}_0$

$e \sim b^2 \sim ba \sim ba^{-1} \sim a^2 \sim ab \sim ab^{-1}$

$b^2 = (ba^{-1})(ab) = (ba \cdot a^{-2})ab = ba \cdot a^{-2} \cdot ab$

$ba^{-1} = ba \cdot a^{-2}$

$ab^{-1} = (ba^{-1})^{-1} = (ba \cdot a^{-2})^{-1} = a^2 (ba)^{-1}$

$\Rightarrow$