1.1.20.

$f_t : X \to X$ $\quad f_0 = f_1 = \mathbb{1}$

homotopy

By lemma 1.19.: $\quad f_{0*} = \beta_{f_t} f_{1*}$ or $\beta_{f_t} = \mathbb{1}$.

$\Rightarrow$ for any loop $[g]$. $\beta_{f_t}[g] = [f_t][g][f_t]^{-1} = [g]$

$\Rightarrow$ $[f_t][g] = [g][f_t]$ i.e. $[f_t]$ is in the center.

1.1.19.

$\gamma : I \to X \ni x_0$

loop $\quad$ 1-dim CW

$\exists$ partition $0 \leq s_0 < s_1 \cdots < s_m = 1$ s.t. $\gamma([s_i; s_{i+1}]) \subset e_\alpha$ (1-cell.

i.e. $\gamma : I \to e_\alpha$ path

Two cases. 1) $\gamma(0) = \gamma(1) = x_0$

e_α is a linear space $\rightsquigarrow$ γ nullhomotopic

2) $\gamma(0) \neq \gamma(1)$ (both endpoints) ?

Let $g : I \to e_\alpha$ be a path, traversing monotonically

$\exists$ homotopy $f_t : I \to I$ between $\mathbb{1}$ and $g^{-1}\gamma$

$\Rightarrow$ $g f_t : I \to e_\alpha$ homotopy $g \simeq \gamma$.

1.2.4

$X = \bigcup_i L_i$, $0 \in L_i \subset \mathbb{R}^3$
lines

$x_0 \in \mathbb{R}^3 \setminus X$

$\mathbb{R}^3 \setminus 0 \xrightarrow[\text{retract}]{\text{deformation}} S^2$

$\mathbb{R}^3 \setminus X \longrightarrow S^2 \setminus (X \cap S^2)$

$$S^n \setminus (X \cap S^2) = S^2 - \{2n \text{ pts}\} \approx \mathbb{R}^3 \setminus X$$

We have $S^2 - \{pt\} \approx \mathbb{R}^2$, thus $\mathbb{R}^3 - X \approx \mathbb{R}^2 - \{2n-1 \text{ pts}\}$

$$\approx \bigvee_{2n-1} S^1$$

$$\Rightarrow \pi_1(\mathbb{R}^3 - X) \approx \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{2n-1}$$

1.2.7.

Y 

x_0 : 0-cell

Two 1-celles  X

Two 2-celles 

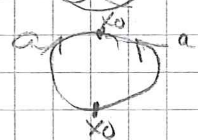
$$\underset{\langle ab \rangle}{\overset{\shortparallel}{N}} \rightarrow \underset{\langle a \rangle * \langle b \rangle}{\pi_1(X, x_0)} \twoheadrightarrow \underset{\mathbb{Z}}{\pi_1(Y, x_0)}$$ (prop 1.26)

Alt

x_0 : 0-cell One 1-cell 

one 2-cell

Remove 1-cell 

Remove 0-cell 

$aa^{-1} = 1$

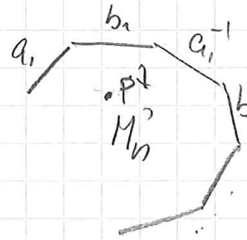
1.2.9

$$S^1$$

Suppose $M_h' \xrightarrow{r} C \xrightarrow{\iota} M_h'$

Induces

$$\begin{tikzcd} \pi_1(C) \arrow[r, "\iota_*"] \arrow[dr] & \pi_1(M_h') \arrow[r, "r_*"] \arrow[d, "ab"] & \pi_1(C) \\ & \pi_1(M_h')_{ab} \arrow[ur] & \end{tikzcd}$$



$[C] \in \pi_1(M_h')$; $[C] = [a_1,b_1]\cdot[a_2,b_2]\cdots$

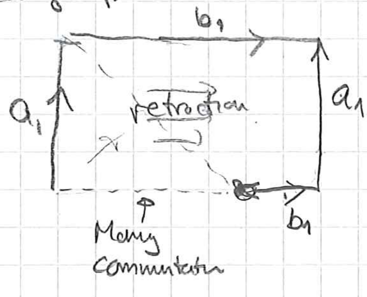
$M_h' - \{pt\} \rightsquigarrow$ boundary

But then

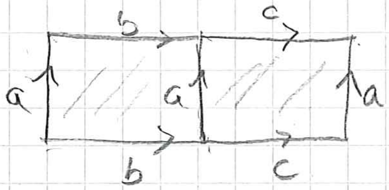
$$ab\, \iota_*[C] = ab([a_1,b_1]\cdots) = \{e\}$$

$\Rightarrow \quad r_* \iota_* [C] = \{e\}$ Contradiction $\Rightarrow$ No retraction

Change picture



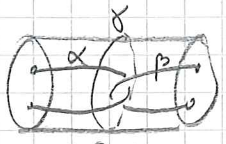
1.2.8.



$\pi_1(X) \simeq \langle a,b,c \mid aba^{-1}b^{-1}, aca^{-1}c^{-1}\rangle$.

i.e. $[a,b]=[a,c]=e$, but $[b,c]\neq e$.

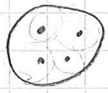
1.2.10.

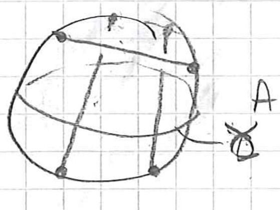


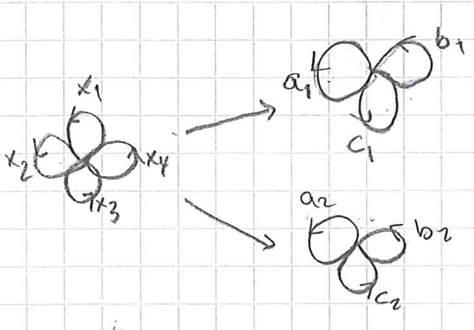
$D^2 \times I$

Van Kampen

$A_1 =$  Two copies

Intersection 

$A_i \cong$  A ball with three arcs removed.



$x_1 \mapsto a_1$, $x_1 \mapsto c_2$

$x_2 \mapsto b_1$, $x_2 \mapsto c_2^{-1}$

$x_3 \mapsto c_1$, $x_3 \mapsto a_2$

$x_4 \mapsto c_1^{-1}$, $x_4 \mapsto b_2$

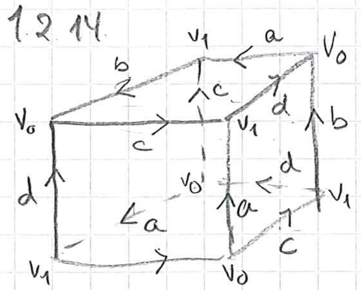
$\Rightarrow$ $a_1 = c_2$, $b_1 = c_2^{-1} = a_1^{-1}$

$a_2 = c_1$, $b_2 = c_1^{-1} = a_2^{-1}$

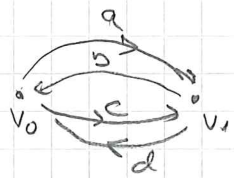
$\pi_1 = \langle a_1, a_2 \rangle$.

$[\gamma] = a a_2 a_1^{-1} a_2^{-1} \neq e$.

1.2.14.



1-cells :



$$\pi_1 \text{ generated by } \overset{\alpha}{ab},\ \overset{\beta}{b^{-1}d},\ \overset{\gamma}{cb}$$

2-cells :

$abcd = (ab)(cb)(b^{-1}d) = 1$ Top $\alpha\gamma\beta = 1$

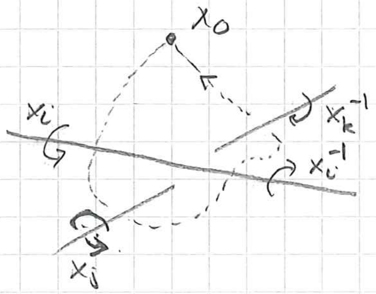
$ac^{-1}d^{-1}b = (ab)(cb)^{-1}(b^{-1}d)^{-1} = 1$ Back $\alpha\gamma^{-1}\beta^{-1} = 1$

$adb^{-1}c^{-1} = (ab)(b^{-1}d)(cb)^{-1} = 1$ Side $\alpha\beta\gamma^{-1} = 1$

$\alpha = (\gamma\beta)^{-1} = \beta\gamma = \gamma\beta^{-1}$ i.e. $\langle \beta, \gamma \ ; \ \beta^2\gamma^2 = 1, \beta\gamma = \gamma\beta^{-1} \rangle$.

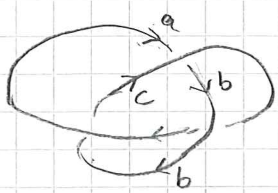
$\beta^{-1}\gamma^{-1} = \beta\gamma = \gamma\beta^{-1}$

1.2.22.



$x_k = x_i x_j x_i^{-1}$

$x_i x_j = x_j x_i \Rightarrow x_j = x_k \quad \underline{\forall j,k}$



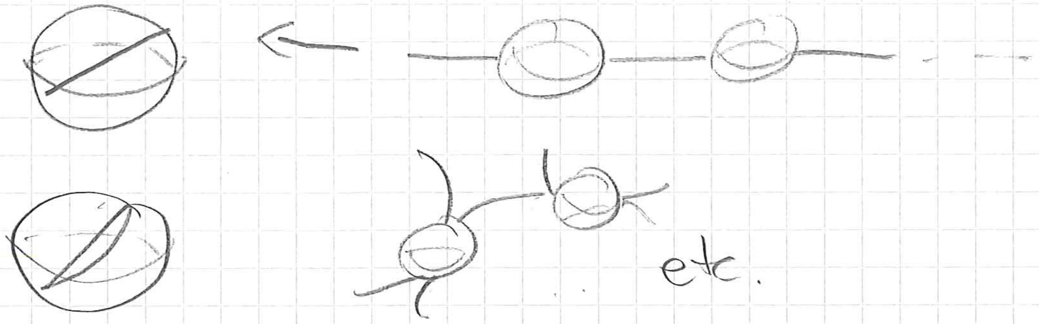
$\langle a,b,c \ ; \ b = cac^{-1}, a = bcb^{-1}, c = aba^{-1} \rangle$

$= \langle a,b \ ; \ b = ab\cancel{a^{-1}a}ab^{-1}a^{-1}, a = b\,aba^{-1}b^{-1} \rangle$

$= \langle a,b \ ; \ ab\,abb^{-1}a^{-1}b^{-1}, aba^{-1}b^{-1}a^{-1}b^{-1} \rangle$

$= \langle a,b \ ; \ abab^{-1}a^{-1}b^{-1} \rangle$.

1.3.4.



1.3.7

 $Y = \{(x, \sin\frac{1}{x}) ; x \in (0, \frac{1}{2\pi}]\} \cup \underbrace{\{(0,y) ; -1 \le y \le 1\}}_{Y_0} \cup C$

$\downarrow$

$S^1 \simeq Y/_{Y_0 \sim pt}$.

We have, suppose
$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ Y \arrow[ur, "\exists \tilde{f}"] \arrow[r, "f"] & S^1 \end{tikzcd}$$
$\tilde{f}([-1,1]) = 0 \in p^{-1}(0)$ bey continuity

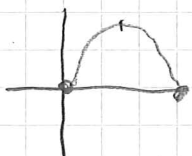
Then $f|_{Y \setminus Y_0}$ is 1-1 $\Rightarrow$ $\tilde{f}|_{Y - Y_0}$ is 1-1

$\Rightarrow$ Define $g : S^1 \to \mathbb{R}$ by

$g(x) = \tilde{f}(x) \quad x \neq 0$

$g(0) = 0$

$\Rightarrow$ g is 1-1.

But  if g non-constant (must by 1-1)

$\exists\, x$ s.t. $g(x) \neq g(0) = g(1)$

$\Rightarrow \exists\, x_1 \neq x_2$ s.t. $g(x_1) = g(x_2)$.

1.3.9.

$f: X \to S^1$ continuous, $x_0 \in X$. $\pi_1(X)$ finite $\Rightarrow f_* \pi_1(X, x_0) \subseteq \pi_1(S^1, 0) \simeq \mathbb{Z}$

finite subgroup $\Rightarrow f_* \pi_1(X, x_0) = \{0\}$

$$\begin{tikzcd} & \tilde{S}^1 = \mathbb{R} \arrow[d, "p"] \\ X \arrow[ur, "\exists \tilde{f}"] \arrow[r, "f"'] & S^1 \end{tikzcd}$$

Prop 1.33. f lifts to $\tilde{f}$

iff $\underset{\{0\}}{\underset{\shortparallel}{f_* \pi_1(X, x_0)}} \subseteq \underset{\{0\}}{\underset{\cap}{p_* \pi_1(\tilde{X}, \tilde{x}_0)}}$

$\pi_1(\mathbb{R}) = 0 \Rightarrow \tilde{f} \simeq [\tilde{x}_0]$.

$\Rightarrow f = p\tilde{f} \simeq p[\tilde{x}_0] = [x_0]$. (null homotopic)

1.3.1.

$$\begin{tikzcd} \tilde{X} \arrow[d, "p"] & \supseteq p^{-1}(A) = \tilde{A} \arrow[d] \\ X & \supseteq A \end{tikzcd}$$

For each pt. $a \in A$, we have a open neighbourhood $a \in U \subseteq X$, such that $p^{-1}(U) = \bigcup U_i$, $U_i \cong U$.

But then look at $U \cap A$, same property

1.3.15

$$\begin{tikzcd} \tilde{X} \arrow[d, "p"] & \supseteq \tilde{X} \arrow[d] \\ X & \supseteq A \end{tikzcd}$$

$$\begin{tikzcd} \pi_1(\tilde{A}) \arrow[r] \arrow[d, "p_*"] \arrow[dr, "0"] & \pi_1(\tilde{X}) = \{0\} \arrow[d, "p_*"] \\ \pi_1(A) \arrow[r, "\iota_*"'] & \pi_1(X) \end{tikzcd}$$

$\Rightarrow p_* \pi_1(\tilde{A}) \subseteq \ker \iota_*$

Suppose $[\gamma] \in \pi_1(A)$ is in $\ker \iota_*$, i.e. $\gamma \underset{\gamma_t}{\simeq} x_0$ in X.

We can lift the homotopy to $\tilde{\gamma} \underset{\tilde{\gamma}_t}{\simeq} [\tilde{x}_0]$. fixes end pts.

$\Rightarrow \tilde{\gamma}$ is a loop.

$\Rightarrow \gamma \in p_* \pi_1(\tilde{A})$

1.3.8.

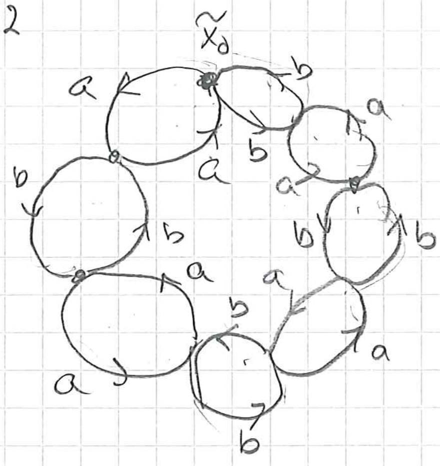
$X \simeq Y$ means $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ $\quad gf \simeq 1\!\!1_X \quad fg \simeq 1\!\!1_Y$

$$\begin{tikzcd}
\tilde{X} \arrow[r, "\exists \tilde{f}"] \arrow[d, "p_X"'] \arrow[dr] & \tilde{Y} \arrow[r, "\exists \tilde{g}"] \arrow[d, "p_Y"] \arrow[dr] & \tilde{X} \arrow[d, "p_X"] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & X
\end{tikzcd}$$

Now

$$\begin{tikzcd}
 & \tilde{X} \arrow[d, "p_X"] \\
\tilde{X} \arrow[ur, "\substack{\text{by uniqueness} \\ 1\!\!1 = \tilde{p}_X}"] \arrow[r, "g f p_X \simeq p_X"'] & X
\end{tikzcd}$$

1.3.12



$\langle a^2, b^2, (ab)^4 \rangle$.

order 8.

$\langle a, b \,;\, a^2, b^2, (ab)^4 \rangle = \{e, a, b, ab, \underset{(ab)^{-1}}{\overset{\shortparallel}{ba}}, aba, bab, abab\}$

Notice $baba = (ab)^{-1}(ab)^{-1} = (abab)^{-1}$

But $e = abababab = (abab)(abab)$, and

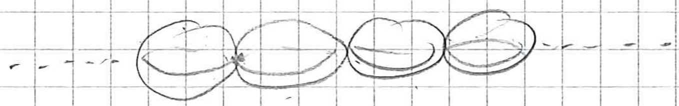
$abab = (abab)^{-1} = baba$.

1.3.14. $\pi_1(\mathbb{RP}^2 \vee \mathbb{RP}^2) \simeq \overset{a}{\mathbb{Z}/(2)} * \overset{b}{\mathbb{Z}/(2)}$

Subgroups gen. by $(ab)^n$ or $(ab)^n, a$

E.g. $\langle (ab)^2, a\rangle \xrightarrow{\simeq} \langle (ab)^2, b\rangle$ $((ab)^2)^{-1} = (ba)^2$; $ababbaba = e$.

$$(ab)^2 \longmapsto (ab)^{-2} = (b^{-1}a^{-1})^2$$
$$a \longmapsto b^{-1}$$



1.3.27.

$$\begin{array}{ccc} \tilde{X} & \supseteq & p^{-1}(x_0) \\ p\downarrow & & \downarrow \\ X & \ni & x_0 \end{array}$$

$[\gamma] \in \pi_1(X, x_0)$ acting on $p^{-1}(x_0)$

unique lifting $\tilde\gamma$ of γ, $\tilde\gamma(0) = \tilde{x}_0$, then

$$[\gamma].\tilde{x}_0 = \tilde\gamma(1).$$

Deck. $\varphi: \tilde{X} \to \tilde{X}$, $p\downarrow \swarrow p$, X

isomorphism

$\varphi(\tilde{x}_0) = \tilde{x}_1$ φ projects to a loop $[\varphi]$ and φ is the unique lifting of $[\varphi] \in \pi_1(X, x_0)$

$\Rightarrow$ Same action by one element, but notice:

$$[\gamma][\gamma'].\tilde{x}_0 = [\gamma\cdot\gamma'].\tilde{x}_0 = (\widetilde{\gamma\cdot\gamma'})(1) =$$

$$\begin{array}{ccc} \tilde{x}_0 \longrightarrow & \tilde\gamma(1) & \\ & \| & \\ & \tilde{x}_1 & \longrightarrow \tilde\gamma'(1) \\ & \| & \\ & \tilde\gamma'(0) & \end{array}$$

$\varphi \sim [\gamma]$, $\varphi' \sim [\gamma']$

$\Rightarrow$ $[\gamma]\cdot[\gamma'] = [\gamma\gamma']$ corresponds to $\varphi\varphi'(\tilde{x}_0)$:

$$\begin{array}{ccc} \tilde{x}_0 \longrightarrow & \tilde\gamma'(1) & \\ & \| & \\ & \tilde{x}_1' & \longrightarrow \tilde\gamma(1) \\ & \| & \\ & \tilde\gamma(0) & \end{array}$$

Anti-isomorphism.