Simplicial homology

$\Delta_n(X)$: free abelian group on n-simplicies $\sigma_\alpha : \Delta^n \to X$, $\sigma_\alpha(\Delta^n) = e_\alpha$

n-chains.

Boundary homomorphism $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha|_{[\ldots \hat{v}_i \ldots]}$$

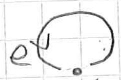
$$\partial_{n-1}\partial_n = 0$$

$$\Downarrow$$

$$\underset{\text{boundaries}}{\mathrm{im}\,\partial_{n-1}} \subseteq \underset{\text{cycles}}{\ker \partial_n}$$

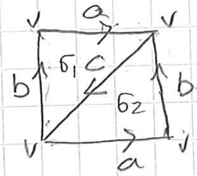
<u>Def</u>. $H_n^\Delta(X) = \ker \partial_n / \mathrm{im}\, \partial_{n-1}$

n'th simplicial homology group.

Ex 1) S^1 ;  $0 \to \mathbb{Z}e \xrightarrow{0} \mathbb{Z}v \to 0$

$\partial e = v - v = 0$

$$\Rightarrow H_n^\Delta(S^1) = \begin{cases} \mathbb{Z} & n = 0, 1 \\ 0 & \text{else} \end{cases}$$

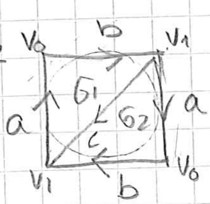
2) Torus. 

$0 \to \mathbb{Z}^2 \to \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z} \to 0$

$\sigma_i \mapsto a + b + c$

$a, b, c \mapsto 0$

$$\Rightarrow H_n^\Delta(T) \simeq \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ 0 & n \geq 3 \end{cases}$$

3) $\mathbb{RP}^2$ 

$0 \to \mathbb{Z}^2 \xrightarrow{\text{injective}} \mathbb{Z}^3 \to \mathbb{Z}^2 \to 0$

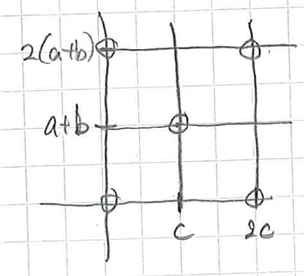
$\sigma_1 \mapsto a + b + c$

$\sigma_2 \mapsto a + b - c$

$a \mapsto v_0 - v_1$

$b \mapsto v_1 - v_0$

$c \mapsto 0$



$\Rightarrow H_2^\Delta(\mathbb{RP}^2) = 0$.

$H_0^\Delta(\mathbb{RP}^2) \simeq \mathbb{Z}$.

$H_n^\Delta(\mathbb{RP}^2) = 0 \quad n \geq 3$

$H_1^\Delta(\mathbb{RP}^2)$: ker gen. by $a + b$, c.

Im gen by $a + b - c$ and $2c$

$\simeq \mathbb{Z}_2$.

Singular homology

Singular n-simplex : $\sigma : \Delta^n \to X$ continuous.

$C_n(X)$: free abelian gp gen by singular n-simplices

Singula n-chains

Notice:

$\sigma|_{[\ldots \hat{v}_i \ldots]} = d_i(\sigma)$

where $d_i : \Delta^{n-1} \to \Delta^n$

$[v_0, \ldots v_{n-1}] \mapsto [v_0, \ldots \hat{v}_i \ldots v_n]$

Boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$

$$\partial_n(\sigma) = \sum_i (-1)^i \, \sigma|_{[\ldots \hat{v}_i \ldots]} \qquad \partial^2 = 0$$

Singular homology group $H_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}$

Prop. $X = \bigsqcup_\alpha X_\alpha$ (disjoint union) $\quad H_n(X) = \bigoplus_\alpha H_n(X_\alpha)$

X_α path-components

Introduce reduced homology

$C_1(X) \to C_0(X) \to C_{-1}(X) \cong \mathbb{Z}$

$\Rightarrow \tilde{H}_n(X) \begin{cases} = H_n(X) & n \neq 0 \\ = H_n(X)/\mathbb{Z} & n = 0 \end{cases}$

Prop. X non-empty path-connected $\Rightarrow H_0(X) \cong \mathbb{Z}$

Pf. $H_0(X) = C_0(X)/\operatorname{im}\partial_1$

Define $\varepsilon : C_0(X) \to \mathbb{Z}$

$\sum n_i \sigma_i \mapsto \sum n_i$

1) - surjective

2) - $\ker \varepsilon = \operatorname{im} \partial_1$

$1 \cdot [v_0, v_1] \mapsto v_1 - v_0$

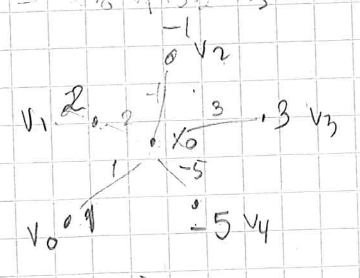
$-1 \, [v_1 v_2] \mapsto v_1 - v_2$

$0 \, [v_2 v_3] \mapsto 0$

$-3 \, [v_3 v_4] \mapsto 3v_3 - 3v_4$

$2 \, [v_4, v_0] \mapsto 2v_0 - 2v_4$

$-v_0 + 2v_1 - v_2 + 3v_3 - 5v_4$



Pf for 2)

$\operatorname{Im}\partial_1 \subset \ker \varepsilon$: $\varepsilon \, \partial_1(\sigma) = \varepsilon\left(\sigma|_{[v_1]} - \sigma|_{[v_0]}\right) = 1 - 1 = 0.$

$\ker \varepsilon \subset \operatorname{Im}\partial_1$: $\varepsilon\left(\sum n_i \sigma_i\right) = 0 \Rightarrow \sum n_i = 0.$

sing. 0-simplex σ: pts

Let $\tau_i : I \to X$; $\tau_i(0) = x_0$ (basept.) $\tau_i(1) = \sigma_i$

View τ_i as a singular 1-simplex $\tau_i \, [v_0, v_1] \to X$, $[v_0, v_1] \cong I$

Then $\partial \tau_i = \sigma_i - x_0$. Hence $\partial\left(\sum n_i \tau_i\right) = \sum n_i \sigma_i - \left(\sum n_i\right) x_0$

$= \sum n_i \sigma_i$

$\Rightarrow \sum n_i \sigma_i \in \operatorname{im}\partial_1$.

$x + 3 - y = 2 \quad x - 1 - y = 3 \quad x - 2 - y = 5$

Prop. $X=\{pt\} \Rightarrow H_n(X) = \begin{cases} \mathbb{Z} & n=0 \\ 0 & n\neq 0 \end{cases}$

Pf. $\exists$ unique singular n-chain : gives

$$\to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

$[p,p] \mapsto 0$

$[p,p,p] \mapsto [p,p]$

etc.

Homotopy Invariance

$f: X \to Y$. induce $f_\#: C_n(X) \to C_n(Y)$

$[\sigma: \Delta^n \to X] \mapsto [f\sigma: \Delta^n \to Y]$

$f_\#(\sigma) = f\sigma$.

Extend by linearity

$\partial(\sigma) = \sum(-1)^i \sigma d_i$

$\Downarrow$

$f_\#\partial(\sigma) = \sum(-1)^i f\sigma d_i$

$\partial f_\# \sigma = \sum(-1)^i f\sigma d_i$

Chain map. $f_\#\partial = \partial f_\#$!!

$\Longrightarrow f_*: H_n(X) \longrightarrow H_n(Y)$

Notice.: $(fg)_* = f_* g_*$ and $1\!\!1_* = 1\!\!1$.

functorial.

Theorem $f \simeq g: X \to Y \Rightarrow f_* = g_*: H_n(X) \to H_n(Y)$

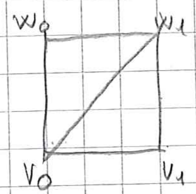
Pf. Sub-divide $\Delta^n \times I$ into simplexes

$\Delta^n \times \{0\} = [v_0, \dots, v_n]$, $\Delta^n \times \{1\} = [w_0, \dots, w_n]$

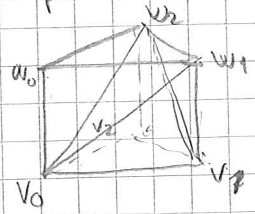
$\Rightarrow \Delta^n \times I = \bigcup_{i=0}^{n} [v_0, \dots, v_i, w_i, \dots, w_n]$

n+1-simplex.

Ex. $n=1$:



$n=2$



Let $F: X \times I \to Y$ homotopy $f \underset{F}{\simeq} g$, define

$$F \circ (\sigma \times 1\!\!1): \Delta^n \times I \to X \times I \to Y$$

Define $P: C_n(X) \to C_{n+1}(Y)$

by $P(\sigma) = \sum_i (-1)^i F\circ(\sigma \times \mathbb{1}) | [v_0,\dots,v_i,w_i,\dots,w_n]$

$$\partial P(\sigma) = \sum_{j\le i} (-1)^{i+j} F\circ(\sigma\times\mathbb{1})|_{[v_0\cdots\hat{v}_j\cdots v_i,w_i\cdots w_n]}$$
$$+ \sum_{j\ge i} (-1)^{i+j+1} F\circ(\sigma\times\mathbb{1})|_{[v_0\cdots v_i w_i\cdots\hat{w}_j\cdots w_n]}$$

$$P\partial(\sigma) = P\Big(\sum_j (-1)^j \sigma|_{[v_0\cdots\hat{v}_j\cdots v_n]}\Big)$$
$$= \sum_{i<j} (-1)^{i+j} F\circ(\sigma\times\mathbb{1})|_{[v_0,\dots,v_i w_i\cdots\hat{w}_j\cdots w_n]}$$
$$+ \sum_{j<i} (-1)^{i+j-1} F\circ(\sigma\times\mathbb{1})|_{[v_0\cdots\hat{v}_j\cdots v_i w_i\cdots w_n]}$$

$$\Rightarrow \partial P(\sigma) + P\partial(\sigma) = \sum_{j=i=0}^{n} (-1)^{i+j} F\circ(\sigma\times\mathbb{1})|_{[v_0,\dots,v_{i-1},w_i,\dots,w_n]}$$
$$+ \sum_{j=i=0}^{n} (-1)^{i+j+1} F\circ(\sigma\times\mathbb{1})|_{[v_0,\dots,v_i,w_{i+1},\dots,w_n]}$$

$$= F\circ(\sigma\times\mathbb{1})|_{[w_0,\dots,w_n]} + F\circ(\sigma\times\mathbb{1})|_{[v_0,w_1,\dots,w_n]} +$$
$$\dots + F\circ(\sigma\times\mathbb{1})|_{[v_0,\dots,w_n]}$$
$$- F\circ(\sigma\times\mathbb{1})|_{[v_0,w_1,\dots,w_n]} - \dots - F\circ(\sigma\times\mathbb{1})|_{[v_0,\dots,v_{n-1},v_n]}$$

$$= F\circ(\sigma\times\mathbb{1})|_{[w_0,\dots,w_n]} - F\circ(\sigma\times\mathbb{1})|_{[v_0,\dots,v_n]}$$
$$= g\circ\sigma - f\circ\sigma = g_\#(\sigma) - f_\#(\sigma)$$

$$\Rightarrow \partial P + P\partial = g_\# - f_\# \qquad P\text{: chain homotopy}$$

$$\begin{tikzcd}
\arrow[r] & C_{n+1}(X) \arrow[r] \arrow[d] & C_n(X) \arrow[r] \arrow[d] \arrow[dl, "P"'] & C_{n-1}(X) \arrow[r] \arrow[d] \arrow[dl, "P"'] & {} \\
\arrow[r] & C_{n+1}(Y) \arrow[r] & C_n(Y) \arrow[r] & C_{n-1}(Y) \arrow[r] & {}
\end{tikzcd}$$

Exact sequence

$$\longrightarrow A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \longrightarrow \cdots$$

$$\underline{\operatorname{im} \alpha_{n+1} = \operatorname{Ker} \alpha_n}.$$

$A \subset X$ subspace $\qquad C_n(X,A) = C_n(X) / C_n(A)$

Induces $\partial_n : C_n(X,A) \to C_{n-1}(X,A)$

$$H_n(X,A) = \operatorname{Ker} \partial_n / \operatorname{Im} \partial_{n-1}$$

Relative homology groups

$$\begin{tikzcd}
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & C_n(X) \arrow[r] \arrow[d] & C_n(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

$\Downarrow$

(1)
$$\begin{tikzcd}
& C_n(A)/\operatorname{Im}\partial_{n+1} \arrow[r] \arrow[d, "\partial_n"] & C_n(X)/\operatorname{Im}\partial_{n+1} \arrow[r] \arrow[d, "\partial_n"] & C_n(X,A)/\operatorname{Im}\partial_{n+1} \arrow[r] \arrow[d, "\partial_n"] & 0 \\
0 \arrow[r] & \operatorname{Ker}\partial_{n-1} \arrow[r] & \operatorname{ker}\partial_{n-1} \arrow[r] & \operatorname{Ker}\partial_{n-1} &
\end{tikzcd}$$

snake lemma $\rightsquigarrow$

$$H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X,A)$$
$$\longrightarrow H_{n-1}(A) \to H_{n-1}(X) \longrightarrow H_{n-1}(X,A).$$

Why (1) right exact:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Im}\partial_{n+1} \arrow[r] \arrow[d, hook] & \operatorname{im}\partial_{n+1} \arrow[r] \arrow[d, hook] & \operatorname{Im}\partial_{n+1} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] & C_n(X) \arrow[r] & C_n(X,A) \arrow[r] & 0
\end{tikzcd}$$

Then snake lemma

Snake lemma

$$\begin{tikzcd}
 & \ker\varphi' \arrow[r, "(1)"] \arrow[d, hook] & \overset{(4)}{\ker\varphi} \arrow[r, "(1)"] \arrow[d, hook] & \overset{(5)}{\ker\varphi''} \arrow[d, hook] \arrow[dddll, bend left, "\delta", "(3)"'] & \\
 & A' \arrow[r, "f"] \arrow[d, "\varphi'"] & A \arrow[r, "g"] \arrow[d, "\varphi"] & A'' \arrow[r] \arrow[d, "\varphi''"] & 0 \\
0 \arrow[r] & B' \arrow[r, "p"] \arrow[d] & B \arrow[r, "q"] \arrow[d] & B'' \arrow[d] & \\
 & \underset{(6)}{\operatorname{coker}\varphi'} \arrow[r, "(2)"] & \underset{(7)}{\operatorname{coker}\varphi} \arrow[r, "(2)"] & \operatorname{coker}\varphi'' &
\end{tikzcd}$$

(1) $x\in\ker\varphi' \Rightarrow \varphi'(x)=0$ ∴ $\varphi(f(x))=p\varphi'(x)=0 \Rightarrow f(x)\in\ker\varphi$

(2) $\bar{y}\in\operatorname{coker}\varphi'$, choose $y\in B'$ ⇝ $\overline{p(y)}\in\operatorname{coker}\varphi$

Another choice $y'\in B'$, $y'-y\mapsto 0\in\operatorname{coker}\varphi' \Rightarrow \exists x\in A'$; $\varphi'(x)=y'-y$

then $\overline{\varphi f(x)}=0\in\operatorname{coker}\varphi$, but $\varphi f(x)=p\varphi'(x)=p(y')-p(y)$

(3) $x''\in\ker\varphi''$, $\varphi''(x'')=0$, choose $x\in A$; $g(x)=x''$. Then $q(\varphi(x))=\varphi''(g(x))=0$

$\Rightarrow \exists y'\in B'$ s.t. $p(y')=\varphi(x)$. Define $\delta(x'')=\overline{y'}$.

choose another $z\in A$, $g(z)=x''$; $g(x-z)=0 \Rightarrow \exists z'\in A'$; $f(z')=x-z$

$\Rightarrow \overline{\varphi'(z')}=0$ and $p\varphi'(z')=\varphi f(z')=\varphi(x-z)=\varphi(x)-\varphi(z)$

Let $y''\in B'$ s.t. $p(y'')=\varphi(z)$. Then

$$p(\varphi'(z'))=p(y')-p(y'')=p(y'-y'')$$

But p is injective $\Rightarrow \varphi'(z')=y'-y'' \Rightarrow \overline{y'}=\overline{y''}$.

Exactness:

(4) $x'\in\ker\varphi' \Rightarrow gf(x')=0$. Suppose $x\in\ker\varphi$; $g(x)=0$, $\Rightarrow \exists x'\in A'$ s.t. $f(x')=x$

$\Rightarrow p(\varphi'(x'))=\varphi f(x')=\varphi(x)=0$ p injective $\Rightarrow \varphi'(x')=0 \Rightarrow x'\in\ker\varphi'$.

(5) $x\in\ker\varphi \Rightarrow \delta g(x)=p(y')$ where $p(y')=\varphi(x)=0$

Suppose $x''\in\ker\varphi''$, $\delta(x'')=0$. Choose $x\in A$; $g(x)=x''$. Let $y'\in B'$ satisfy $p(y')=\varphi(x)$. Then $y'=\varphi'(x')$ for some $x'\in A'$

$\Rightarrow \varphi f(x')=p\varphi'(x')=p(y')=\varphi(x)$. Thus $-f(x')+x\in\ker\varphi$

and $g(-f(x')+x)=-gf(x')+g(x)=x''$.

(6) $x'' \in \ker \varphi''$, choose $x \in A$, $g(x) = x''$, and $y' \in B'$ s.t
$p(y') = \varphi(x)$. Then $p\delta(x'') = p(y') = \varphi(x) \longmapsto 0$ in $\operatorname{coker} \varphi$.
Let $y' \in B'$ s.t. $p(y') = \varphi(x)$ for some $x \in A$. Then.
$\varphi'' g(x) = q\varphi(x) = qp(y') = 0$ and $g(x) \in \ker \varphi''$, But $\overline{y'} = \delta(g(x))$

(7) $y' \in B' \Rightarrow qp(y') = 0$.
Let $y \in B$ s.t $q(y) = \varphi''(x'')$. choose $x \in A$ s.t $g(x) = x''$
Then $q(y - \varphi(x)) = q(y) - q\varphi(x) = \varphi''(g(x)) - \varphi'' g(x) = 0$
Thus $\exists\, y'$ s.t. $p(y') = y - \varphi(x)$, But $\overline{p(y')} = \overline{y}$.

Example. $\partial D^n \subset D^n$

$$\to H_m(\partial D^n) \to H_m(D^n) \to H_m(D^n, \partial D^n) \to H_{m-1}(\partial D^n) \to \cdots$$

gives. $H_m(D^n, \partial D^n) \simeq H_{m-1}(\partial D^n) \simeq H_{m-1}(S^{n-1}) \quad m \geq 1$

$$\text{i.e. } H_m(D^n, \partial D^n) = \begin{cases} \mathbb{Z} & m = n \\ 0 & \text{else} \end{cases}$$

$$0 \to H_1(D^n, \partial D^n) \to \mathbb{Z} \xrightarrow[\text{Iso}]{} \mathbb{Z} \to H_0(D^n, \partial D^n)$$

$$\Rightarrow \quad H_1(D^n, \partial D^n) = H_0(D^n, \partial D^n) = 0.$$

For $n = 1$.

$$0 \to \underset{\overset{\parallel}{0}}{H_1(\partial D^1)} \to \underset{\overset{\parallel}{0}}{H_1(D^1)} \to H_1(D^1, \partial D^1) \to \underset{\simeq \mathbb{Z}^2}{H_0(\partial D^1)}$$

$$\Leftarrow$$

$$\to \underset{\overset{\parallel}{\mathbb{Z}}}{H_0(D^1)} \to H_0(D^1, \partial D^1) \to 0$$

Example.

$\{x_0\} \subseteq X.$

$$\underset{\overset{\parallel}{0}}{H_n\{x_0\}} \to H_n(X) \to H_n(X, \{x_0\}) \to \underset{\overset{\parallel}{0}}{H_{n-1}\{x_0\}}$$

$$\underset{\overset{\parallel}{0}}{H_1(x_0)} \to H_1(X) \to H_1(X, x_0) \to \underset{\overset{\cong}{\mathbb{Z}}}{H_0(x_0)} \xrightarrow{\simeq} \underset{\overset{\cong}{\mathbb{Z}}}{H_0(X)} \to \underset{\overset{\parallel}{0}}{H_0(X, x_0)} \to 0$$

$$\begin{array}{ccc} f: X & \longrightarrow & Y \\ \cup\!\!\!\mid & & \cup\!\!\!\mid \\ A & \longrightarrow & B \end{array} \quad \rightsquigarrow \quad f_\#: C_n(X, A) \to C_n(Y, B)$$

$$f_\# \partial = \partial f_\#$$

$$\rightsquigarrow f_*: H_n(X, A) \to H_n(Y, B)$$

Prop. $f \simeq g : (X, A) \to (Y, B) \Rightarrow f_* = g_* : H_n(X, A) \to H_n(Y, B)$

Notice. $B \subset A \subset X$: $\quad 0 \to C_n(A, B) \to C_n(X, B) \to C_n(X, A) \to 0$

$$\underset{\text{induces}}{\rightsquigarrow} \quad \cdots \to H_n(A, B) \to H_n(X, B) \to H_n(X, A) \to H_{n-1}(A, B) \to \cdots$$