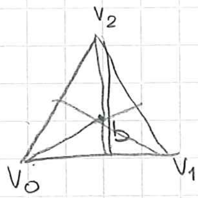
1) Barycentric subdivision of Simplices.

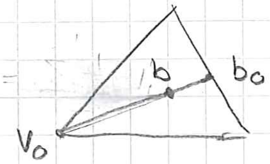
$[v_0, .., v_n]$ n-simplex $\qquad b = \sum_{i=0}^{n} \frac{1}{n+1} v_i$ barycenter



equilateral triangle

by symmetry the faces are divided as they should

Diameter of subdivision



$b_0 = \frac{1}{n}(v_1 + \cdots + v_n)$

$b = \frac{1}{n+1}(v_0 + \cdots + v_n)$

$$|v_0 - b| = \left|v_0 - \frac{1}{n+1}(v_0 + \cdots + v_n)\right| = \left|v_0 - \frac{1}{n+1} v_0 - \frac{n}{n+1} b_0\right| = \frac{n}{n+1}|v_0 - b_0| \leq \frac{n}{n+1}|v_i - v_j|$$

$\Rightarrow$ Iterated subdivision gives diameter $\left(\frac{n}{n+1}\right)^r \to 0$.

2) Linear chains: $Y \subseteq \mathbb{R}^m$, convex set: $\sigma : \Delta^n \to Y$ linear map.

determined by the image w_i of vertices

$$\partial : LC_n(Y) \to LC_{n-1}(Y)$$

Reduced: $LC_{-1}(Y) = \mathbb{Z}$ generated by the empty simplex $\{\emptyset\}$

$\partial[v] = \{\emptyset\}$.

Let $b \in Y$. Then b defines homomorphism

$$b : LC_n(Y) \to LC_{n+1}(Y)$$

$$[w_0, .., w_n] \mapsto [b, w_0 .. , w_n] \quad \underline{\text{cone}}.$$

$$\partial b[w_0, .., w_n] = [w_0, .., w_n] - b\,\partial[w_0, .., w_n]$$

Extended by linearity: $\partial b(\alpha) = \alpha - b\partial(\alpha)$

or $\partial b + b\partial = \mathbb{1}$.

chain homotopy !!

Define subdivision homomorph $S : LC_n(Y) \to LC_n(Y)$

Inductively: $\lambda : \Delta^n \to Y \in LC_n(Y)$, $b_\lambda = \lambda(b)$ ← barycenter

Then $S(\lambda) = b_\lambda S(\partial\lambda)$

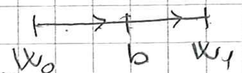
Starting point $S[\emptyset] = [\emptyset]$

Notice

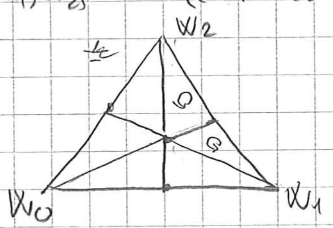
$$S[w_0] = w_0(S\partial[w_0]) = w_0(S[\emptyset]) = w_0[\emptyset]) = [w_0]$$

$b = \frac{w_0+w_1}{2}$ $\quad S[w_0,w_1] = b(S([w_1]-[w_0])) = b([w_1]-[w_0])$

$$= [b,w_1] - [b,w_0]$$



$b = \frac{w_0+w_1+w_2}{3}$ $\quad S[w_0,w_1,w_2] = b\, S([w_1,w_2] - [w_0 w_2] + [w_0,w_1])$



Notice $\partial S\lambda = \partial b_\lambda(S\partial\lambda) =$

$$= (\mathbb{1} - b_\lambda\partial)(S\partial\lambda)$$

$$= S\partial\lambda - b_\lambda \partial S(\partial\lambda) \qquad \text{by induction on } n$$

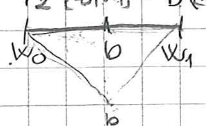
$$= S\partial\lambda - b_\lambda S\partial\partial\lambda = S\partial\lambda$$

since $\partial S[w_0] = [\emptyset]$
$S\partial[w_0] = [\emptyset]$.

$$\begin{tikzcd}
\to LC_2(Y) \arrow[r] \arrow[d, shift right, "S"'] \arrow[d, shift left, "id"] & LC_1(Y) \arrow[r] \arrow[d, shift right, "S"'] \arrow[d, shift left, "id"] \arrow[dl, "T_1"] & LC_0(Y) \arrow[r] \arrow[d, shift right, "S"'] \arrow[d, shift left, "id"] \arrow[dl, "T_0"] & LC_{-1}(Y) \arrow[r] \arrow[d, shift right, "S"'] \arrow[d, shift left, "id"] \arrow[dl, "T_{-1}"] & 0 \\
\to LC_2(Y) \arrow[r] & LC_1(Y) \arrow[r] & LC_0(Y) \arrow[r] & LC_{-1}(Y) \arrow[r] & 0
\end{tikzcd}$$

$T_1[w] = [w,w]$

$T_2[w_0,w_1] = b([w_0,w_1] - [w_1,w_1] + [w_0,w_0])$



Define T inductively $T_{-1} = 0$.

$$T_n(\lambda) = b_\lambda(\lambda - T_{n-1}\partial\lambda)$$

Notice $\partial T + T\partial = \mathbb{1} - S$ ok for $n = -1$

Induction: $\partial T_n \lambda = \partial b_\lambda(\lambda - T_{n-1}\partial\lambda)$

$$= \lambda - T_{n-1}\partial\lambda - b_\lambda \partial(\lambda - T_{n-1}\partial\lambda)$$

$$= \lambda - T_{n-1}\partial\lambda - b_\lambda(\partial\lambda - \partial T_{n-1}\partial\lambda)$$

$\partial T_{n-1} + T_{n-2}\partial = \mathbb{1} - S$

$$= \lambda - T_{n-1}\partial\lambda - b_\lambda\left(S\partial\lambda + T_{n-2}(\partial(\partial\lambda))\right)$$

or $\mathbb{1} - \partial T_{n-1} = S + T_{n-2}\partial$

$$= \lambda - T_{n-1}\partial\lambda - \underbrace{b_\lambda S\partial\lambda}_{S\lambda}$$

$$= \lambda - T_{n-1}\partial\lambda - S\lambda$$

$$\Rightarrow \quad \partial T_n + \underline{T_{n-1}}\partial = \mathbb{1} - S.$$

3) General case

Define $\quad S : C_n(X) \to C_n(X)$

by $\quad S\sigma = \sigma_\# S\Delta^n \qquad \sigma : \Delta^n \to X$ singular n-simplex

Notice $\quad \partial S\sigma = \partial\sigma_\# S\Delta^n = \sigma_\#\partial S\Delta^n = \sigma_\# S\partial\Delta^n$

$$= \sigma_\# S\Big(\sum_i (-1)^i \underset{\text{face}}{\Delta^n_i}\Big) = \sum_i (-1)^i \sigma_\# S\Delta^n_i$$

$$= \sum_i (-1)^i S(\sigma|_{\Delta^n_i}) = S\Big(\sum_i (-1)^i \sigma|_{\Delta^n_i}\Big) = S\partial\sigma$$

Define $\quad T : C_n(X) \to C_{n+1}(X)$

by $\quad T\sigma = \sigma_\# T\Delta^n$

Notice $\quad \partial T\sigma = \partial\sigma_\# T\Delta^n = \sigma_\#\partial T\Delta^n$

$$= \sigma_\#(\Delta^n - S\Delta^n - T\partial\Delta^n)$$

$$= \sigma - S\sigma - \sigma_\# T\partial\Delta^n$$

$$= \sigma - S\sigma - T(\partial\sigma).$$

(4) Iteration.

chain homotopy $D_m = \sum_{i=0}^{m-1} TS^i$ between $\mathbb{1}$ and S^m;

$$\partial D_m + D_m \partial = \sum_{i=0}^{m-1} \partial TS^i + TS^i \partial$$
$$= \sum_{i=0}^{m-1} (\partial TS^i + T\partial S^i)$$
$$= \sum_{i=0}^{m-1} (\partial T + T\partial)\, S^i$$
$$= \sum_{i=0}^{m-1} (\mathbb{1} - S)\, S^i = \sum_{i=0}^{m-1} S^i - S^{i+1} = S^0 - S^m$$
$$= \mathbb{1} - S^m$$

X: $\mathcal{U} = \{U_j\}$

subspaces such that interior form an open cover

$C_n^{\mathcal{U}}(X) \subset C_n(X)$

$\sum n_i \sigma_i$ such that $\operatorname{im} \sigma_i \subset U_j$ for some j.

chain complex

$\longrightarrow H_n^{\mathcal{U}}(X)$

Prop. $H_n^{\mathcal{U}}(X) \approx H_n X$

via $C_n^{\mathcal{U}}(X) \underset{\iota}{\overset{\rho}{\rightleftarrows}} C_n(X)$ ← chain map

$\iota\rho,\ \rho\iota \simeq \mathbb{1}$.

$\sigma: \Delta^n \longrightarrow X$ $\rightsquigarrow$ $S^m\sigma \in C_n^{\mathcal{U}}(X)$ for some $m = m(\sigma)$

n-simplex

Define $D: C_n(X) \longrightarrow C_{n+1}(X)$

$\sigma \longmapsto D_{m(\sigma)}\sigma$

Put $\rho\sigma = \sigma - \partial D\sigma - D\partial\sigma$

$$\Rightarrow \partial\rho(\sigma) = \partial\sigma - \partial^2 D\sigma - \partial D\partial\sigma = \partial\sigma - \partial D\partial\sigma$$
$$\rho\partial(\sigma) = \partial\sigma - \partial D\partial\sigma - D\partial^2\sigma = \partial\sigma - \partial D\partial\sigma.$$

Notice

$$p(\sigma) = \sigma - \partial D\sigma - D(\partial\sigma)$$

$$= \sigma - \partial D_{m(\sigma)}\sigma - D(\partial\sigma)$$

$\partial D_m + D_m \partial = \mathbb{1} - S^m$

$$= \underset{\substack{\Uparrow \\ C_n^{\mathcal{U}}(X)}}{S^{m(\sigma)}\sigma} + \underbrace{D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma)}$$

linear combinations of

$$D_{m(\sigma)}(\sigma_j) - D_{m(\sigma_j)}(\sigma_j)$$

$\sigma_j = \sigma|_{face}$, thus $m(\sigma_j) \leq m(\sigma)$

hence of term $TS^i(\sigma_j)$ with $i \geq m(\sigma_j)$
$\in C_n^{\mathcal{U}}(X)$

Notice by definition $\partial D + D\partial = \mathbb{1} - p$

$p c = c - \partial D c + D\partial c = c \quad D_0 = 0$ on $C_n^{\mathcal{U}}(X)$

QED.

Excision Theorem

$$\left(\begin{array}{l} Z \subset A \subset X \\ \overline{Z} \subset \mathring{A} \\ \text{closure interior} \end{array} \right)$$

$A, B \subset X$, $\mathring{A} \cup \mathring{B} = X$

$$\Rightarrow H_n(B, A\cap B) \cong H_n(X, A)$$

Pf.

Cover $\mathcal{U} = (A, B)$ Notation $C_n(A+B) = C_n^{\mathcal{U}}(X)$

By long argument:

$$C_n(A+B)/C_n(A) \hookrightarrow C_n(X)/C_n(A)$$

induces iso in homology

$$C_n(B)/C_n(A\cap B) \longrightarrow C_n(A+B)/C_n(A)$$

induce iso.

QED.

Prop

$A \subset X$ $\qquad q : (X, A) \longrightarrow (X/A, A/A)$

A deformation retract of some neighbourhood in X $\qquad$ induces iso

$$q_* : H_n(X, A) \longrightarrow H_n(X/A)$$

Pf. $A \subseteq V \subset X$ (retracts)

$$\begin{tikzcd}
H_n(X,A) \arrow[r, "\simeq", "\text{iso (retract)}"'] \arrow[d, "q_*"', "\text{iso}"] & H_n(X,V) \arrow[r, "\simeq", "\text{iso (excision)}"'] \arrow[d, "q_*"] & H_n(X-A, V-A) \arrow[d, "q_*", "\approx \text{ iso (outside of } A)"'] \\
H_n(X/A, A/A) \arrow[r, "\simeq", "\text{iso (retract)}"'] & H_n(X/A, V/A) \arrow[r, "\simeq", "\text{iso (excision)}"'] & H_n(X/A - A/A, V/A - A/A)
\end{tikzcd}$$

$\Rightarrow$ iso q_*

Def.

$A \subset X$ $\qquad (X, A)$ <u>Good pair</u>

- $\neq \emptyset$
- closed subspace
- deformation retract of neighbourhood in X

Ex.

Each point $x \in X$ (CW-complex) has arbitrary small contractible open neighbourhoods

Ex $\quad U \subseteq \mathbb{R}^m$ open $\quad V \subseteq \mathbb{R}^n$ open $\quad U \approx V \Rightarrow m = n$.

Pf $\quad H_k(U, U \setminus \{x\}) \approx H_k(\mathbb{R}^m, \mathbb{R}^m - \{x\})$

$$H_k(\mathbb{R}^m, \mathbb{R}^m - \{x\}) \approx \tilde{H}_{k-1}(\mathbb{R}^m - \{x\}) \approx \tilde{H}_{k-1}(S^{m-1})$$

$\neq 0$ if $k \neq m$.

$\square$