Degree $f: S^n \to S^n$ $n>0$ induce $f_*: H_n(S^n) \to H_n(S^n)$

$$\begin{array}{ccc} H_n(S^n) & \xrightarrow{f_*} & H_n(S^n) \\ \cong & & \cong \\ \mathbb{Z} & & \mathbb{Z} \\ 1 & \longmapsto & d \end{array}$$

Dedn. $d = \deg(f)$ degree.

Properties.

1) $\deg \mathbb{1} = 1$.

2) $\deg(f) \neq 0 \Rightarrow f$ surjective

Let $x_0 \in S^n \setminus f(S^n) \neq \emptyset$. Then.

$$\begin{tikzcd} S^n \arrow[r] \arrow[rr, bend right, "f"'] & S^n \setminus \{x_0\} \arrow[r, hook] & S^n \end{tikzcd}$$ possible to factorize f.

$$\Rightarrow \begin{tikzcd} H_n(S^n) \arrow[r] \arrow[rr, bend right, "f_*"'] & H^n(S^n \setminus \{x_0\}) = 0 \arrow[r] & H_n(S^n) \end{tikzcd}$$

$\Rightarrow f_* = 0$.

3) $f \simeq g \Rightarrow f_* = g_* \Rightarrow \deg(f) = \deg(g)$

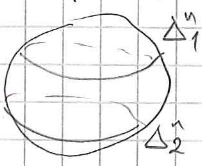
Opposite: Hopf (1925)

4) $\deg(fg) = \deg(f) \cdot \deg(g)$ since $(fg)_* = f_* g_*$

$\Rightarrow$ (f homotopy equivalence $\Rightarrow \exists g$ s.t. $fg \simeq \mathbb{1}$

$\Rightarrow \deg(f) \cdot \deg(g) = 1 \Rightarrow \deg(f) = \pm 1$)

5) f reflection on $S^n \Rightarrow \deg(f) = -1$.



Generator: $\Delta_1^n - \Delta_2^n$

Reflection: $\Delta_1^n - \Delta_2^n \mapsto \Delta_2^n - \Delta_1^n = -(\Delta_1^n - \Delta_2^n)$

6) $-\mathbb{1}: S^n \to S^n \Rightarrow \deg(-\mathbb{1}) = (-1)^{n+1}$
$x \mapsto -x$.

$-\mathbb{1}: (x_0, \dots, x_n) \mapsto (-x_0, \dots, x_n) \mapsto \dots \dots \to (-x_0, -x_1, \dots, -x_n)$

$\deg(-\mathbb{1}) = (-1) \cdot (-1) \cdots (-1) = \underline{\underline{(-1)^{n+1}}}$

7) $f: S^n \to S^n$ fix point free $\Rightarrow \deg f = (-1)^{n+1}$

Pf. Suppose $f(x) \neq x$. Then the line segment from $f(x)$ to $-x$ given by $t \mapsto (1-t)f(x) - tx = \gamma(t)$ $0 \le t \le 1$ we have $\gamma(t) \neq 0$ $\forall t$.

Why? $\gamma(t) = 0 \Leftrightarrow (1-t)f(x) - t(x) = 0$

or $f(x) = \frac{t}{1-t} x$ i.e. for $t = \frac{1}{2}$

we have $f(x) = \frac{\frac{1}{2}}{1-\frac{1}{2}} x = x$ fix pt.

$\Rightarrow$ If $f(x) \neq x$ $\forall x$, then

$$f_t(x) = \frac{(1-t)f(x) - tx}{\|(1-t)f(x) - tx\|}$$

defines homotopy between $f(x)$ and $f_1(x) = -x$.

$\Rightarrow f_* = f_{1*}$ and $\deg f_1 = (-1)^{n+1}$

$\Rightarrow \underline{\underline{\deg f = (-1)^{n+1}}}$

Applications.

S^n has a continous field of nonzero tangents iff n is odd

Pf.

Let $v(x)$ be a vector field on S^n. Then $v(x) \perp x$.

If $v(x) \neq 0$ we can normalize; $\frac{v(x)}{|v(x)|}$

Consider $f_t(x) = \cos t \cdot x + \sin t \cdot \frac{v(x)}{|v(x)|}$

$$\Rightarrow \|f_t(x)\|^2 = f_t(x) \cdot f_t(x) = \cos^2 t\,(x \cdot x) + 2\cos t \sin t\; x \cdot \frac{v(x)}{|v(x)|} + \sin^2 t \cdot \frac{v(x) \cdot v(x)}{|v(x)|^2}$$

$$= \cos^2 t + \sin^2 t = \underline{1}.$$

$\Rightarrow f_t(x) \in S^n$

Homotopy between $f_0(x) = x$ and $f_\pi(x) = -x$

$\Rightarrow \deg \mathbb{1} = \deg f_\pi = (-1)^{n+1}$

$\Rightarrow n+1 \equiv 0 \pmod 2$ i.e. $n \equiv 1 \pmod 2$

If $n \equiv 1 \pmod 2$, then define

$$V(x_0, \ldots, x_n) = (x_1, -x_0, x_3, -x_2, \ldots, x_n, x_{n-1})$$

We have $x \cdot V(x) = 0$ and $|V(x)| = 1$.

Notice that if n is even, the only non-trivial gp which can act freely on S^n is $\mathbb{Z}_2$.

$\underset{\text{acts freely}}{g \in G} \Rightarrow g : X \to X$ fix pt. free homeomorphism

$\Rightarrow \deg g = \pm 1 \Rightarrow \deg : G \to \mathbb{Z}_2$ homeomorphism.

in fact $\deg g = (-1)^{n+1}$.

If $n \equiv 0 \pmod 2$, then $\deg(g) = -1 \ \forall g \neq e$.

$\Rightarrow \ker(\deg) = \{e\}$ and $G \cong \mathbb{Z}_2$.

Local degree

$f : S^n \to S^n \ni y$, $f^{-1}(y) = \{x_1, \ldots, x_m\}$.

Let $x_i \subset \underset{\text{open ngbh}}{U_i} \subset X$, $f(U_i) \subset \underset{\text{ngbh open}}{V} \subset Y$

$\Rightarrow f(U_i - \{x_i\}) \subset V - \{y\}$.

$$\begin{tikzcd}
 & H_n(U_i, U_i - \{x_i\}) \arrow[r, "f_*"] \arrow[dl, "\cong"', "\text{excision}"] \arrow[d, "k_i"] & H_n(V, V-\{y\}) \arrow[d, "\cong", "\text{excision}"'] \\
H_n(S^n, S^n - \{x_i\}) & H_n(S^n, S^n - f^{-1}(y)) \arrow[l, "p_i"'] \arrow[r, "f_*"] & H_n(S^n, S^n - \{y\}) \\
 & H_n(S^n) \arrow[ul, "\cong"] \arrow[u, "j"] \arrow[r, "f_*"] & H_n(S^n) \arrow[u, "\cong"']
\end{tikzcd}$$

local degree $\deg f|_{x_i}$

Prop. $$\deg f = \sum_i \deg f|_{x_i}$$

Pf. $$H_n(S^n, S^n - f^{-1}(y)) \approx \bigoplus H_n(U_i, U_i - \{x_i\})$$
by excision. $\qquad H_n(U_i, U_i - \{x_i\}) \cong \mathbb{Z}$.

k_i : inclusion of the i'th summand

p_i : projection onto — " —

Diagram

$$\begin{tikzcd}
 & \mathbb{Z} \arrow[r, "f_*"] \arrow[d, "k_i"] \arrow[dl] & \mathbb{Z} \arrow[d] \\
\mathbb{Z} & \mathbb{Z}^m \arrow[l] \arrow[r, "f_*"] & \mathbb{Z} \\
 & \mathbb{Z} \arrow[u, "j"] \arrow[r, "f_*"] \arrow[ul] & \mathbb{Z} \arrow[u]
\end{tikzcd}$$

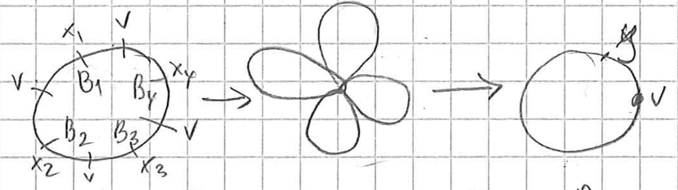
$1 \mapsfrom 1$

1) $j(1) = (1, \dots, 1) \in \mathbb{Z}^m$
$= \sum_i k_i(1)$

2) $f_*(k_i(1)) = \deg f|_{x_i}$
$\Downarrow$
$f_*\left(\sum_i k_i(1)\right) = \sum_i \deg f|_{x_i}$

3) $f_* j(1) = f_*\left(\sum_i k_i(1)\right) = \sum \deg f|_{x_i}$
$\shortparallel$
$f_*(1) = \deg f$.

Ex.



$$\begin{tikzcd}
S^n \arrow[r, "q"] \arrow[rr, bend right, "f = pq"'] & \bigvee_k S^n \arrow[r, "p"] & S^n
\end{tikzcd}$$

$f^{-1}(y) = \{x_1, \dots, x_m\}$

$\mathring{B}_i \to S^n - \{y\}$ homeomorphism $\Rightarrow \deg f|_{x_i} = \pm 1$

By orienting the summands of $\vee S^n$ can assume $\deg f|_{x_i} = +1 \ \forall i$ (or $-1\ \forall i$)

$$\Rightarrow \underline{\underline{\deg f = \pm k.}}$$

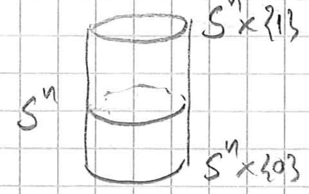
Ex. $f: S^1 \to S^1$, $z \mapsto z^k$ $\quad f^{-1}(y) = \{x_1, \dots, x_k\}$.

Stretch ngh of x_i s.t. f locally is rotation. $\Rightarrow \deg f|_{x_i} = 1$

$\Rightarrow$ deg $f = k$.

Ex. $f: S^n \to S^n$, suspension $Sf: S^{n+1} \to S^{n+1}$

then $\deg Sf = \deg f$.



$CS^n = S^n \times I / S^n \times \{1\}$ cone

$S^{n+1} = SS^n = CS^n / S^n \times \{0\}$ suspension

$$\begin{tikzcd}
\tilde{H}_{n+1}(S^{n+1}) \arrow[r, "\delta", "\text{iso}"'] \arrow[d, "Sf_*"] \arrow[dr, phantom, "\circ"] & \tilde{H}_n(S^n) \arrow[d, "f_*"] \\
\tilde{H}_{n+1}(S^{n+1}) \arrow[r, "\delta", "\text{iso}"'] & \tilde{H}_n(S^n)
\end{tikzcd}$$

$S^n \subset CS^n$

$$\to H_{n+1}(S^n) \to \underset{\substack{\| \\ 0}}{H_{n+1}(CS^n)} \to \underset{\substack{\cong \\ H_{n+1}(S^{n+1})}}{H_{n+1}(CS^n, S^n)} \xrightarrow{\delta} H_n(S^n) \to \underset{\substack{\| \\ 0}}{H_n(CS^n)} \to$$