

1. Suppose $f: D^n \rightarrow D^n$ fix pt free.

Let $p: S^n \rightarrow S^n$ reflection through equatorial plane

Define $g: S^n \rightarrow S^n; g(x) = \begin{cases} f(p(x)) & x \in S_+^n \cong D^n \\ f(x) & x \in S_-^n \end{cases}$

$\Rightarrow g$ is continuous and fix pt free

Define homotopy $F(x,t) = \frac{(1-t)g(x) + t(-11)}{11}$ well-defined \Leftarrow fix pt free

$\Rightarrow g \simeq -11$ and $\deg(g) = \deg(-11) = \pm 1$

But g is not surjective $\Rightarrow \deg(g) = 0$ Contradiction

2.

$f: S^{2n} \rightarrow S^{2n}$

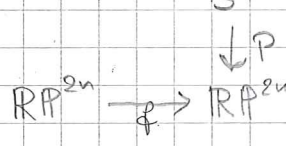
a) 1) If $f(x) \neq x \forall x$, then $f \simeq -1$ and $\deg(f) = (-1)^{2n-1} = -1$

2) If $f(x) \neq -x \forall x$, then $-f(x) \neq x$ and $\deg(-f) = (-1)^{2n-1} = -1$

$\Rightarrow \deg(f) = 1$

$\Rightarrow f(x) \neq x$ and $f(x) = -x$ is impossible

b) We have $S^{2n}/x \sim -x \cong \mathbb{R}P^{2n}$ (universal covering)



By lifting criterion $\exists g: \mathbb{R}P^{2n} \rightarrow S^{2n}$ s.t. $pg = f$

Compose with $p: S^{2n} \rightarrow \mathbb{R}P^{2n}$. Then $\exists y \in S^{2n}$ s.t.

either $gp(y) = y$ or $gp(y) = -y$. But in either cases

$f(p(y)) = pgp(y) = p(\pm y) = p(y)$

$\Rightarrow f$ has a fix pt.

c) $T: \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ without eigenvectors.

T induces map $\bar{T}: \mathbb{R}P^{2n-1} \rightarrow \mathbb{R}P^{2n-1}$ by

$\bar{T}([v]) = [T(v)]$

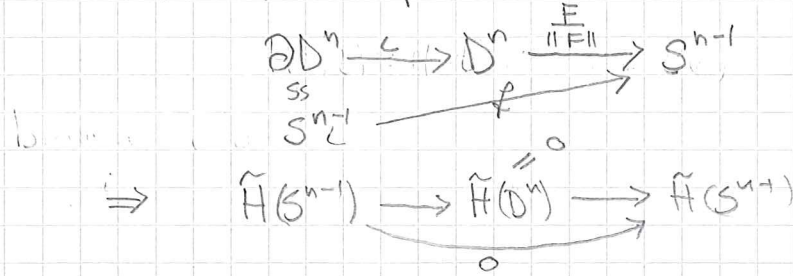
Then \bar{T} is fix pt free since $\bar{T}([v]) = [v]$ is equiv

to $T(v) = \lambda v$ for some $\lambda \in \mathbb{R} \Rightarrow T$ has eigenvector.

3. $f: S^n \rightarrow S^n \Rightarrow f$ has a fix pt; $x \in S^n, f(x) = x$
 $\deg(f) = 0$ But also $-f$ has a fix pt; $y \in S^n -f(y) = y$ or $f(y) = -y$

F vector field on D^n with $F(x) \neq 0 \forall x$.

Let Consider the composition



$\Rightarrow \deg(f) = 0 \Rightarrow \exists x \in \partial D^n$ s.t. $f(x) = x$

But then $F(x) = \|F(x)\| \cdot x$ i.e. points radially outward

Similar for $f(y) = -y$.

4. Construction of surjective map $f: S^n \rightarrow S^n$ of degree 0.

Let $f: S^1 \rightarrow S^1$ be given by $f = \mathbb{1}_{S^1} \cdot \overline{\mathbb{1}}_{S^1}$

- surjective.
- nullhomotopic $\Rightarrow \deg(f) = \deg(0) = 0$.

But $\deg Sf = \deg f$ and suspension preserves nullhomotopy.

7. $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ Elementary row operations E

invertible linear transformation

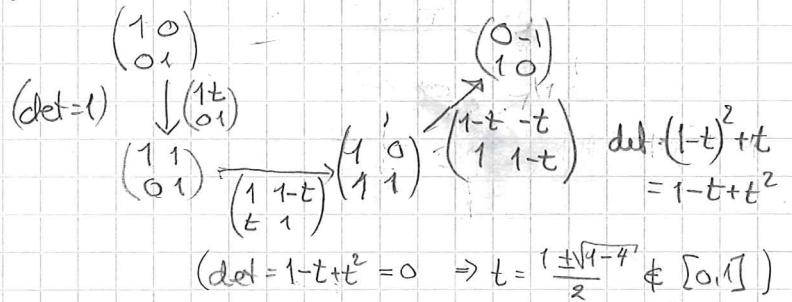
choose basis: $M(f) = A$

- Scaling a row: $E_t = t \cdot I + (1-t)E$ positive } non-sg
- $E_t = t(-I) + (1-t)E$ negative }

• changing two rows (and change sign of one)

• Adding multiple of one row

$E_t = t \cdot I + (1-t)E$ nonsing



Composition gives homotopy

$E_t: E_0 = A$
 $E_1 = \text{diag}(\pm 1)$

$\Rightarrow \det A$ and $\det E_1$ has same sgn.
 (all operations preserve sgn of det)

Can assume A is diagonal with entries ± 1 .

$$\begin{array}{ccccc} H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) & \xrightarrow{\cong} & H_{n-1}(\mathbb{R}^n - \{0\}) \simeq H_{n-1}(S^{n-1}) \simeq \mathbb{Z} \\ \downarrow A_* & & \downarrow & & \downarrow \\ H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) & \xrightarrow{\cong} & H_{n-1}(\mathbb{R}^n - \{0\}) \simeq H_{n-1}(S^{n-1}) \simeq \mathbb{Z} \end{array}$$

We have $f = f_1 \cdots f_m$, f_i reflection through a coordinate hyperpl
 $\Rightarrow \deg(f) = (-1)^m = \det(A) = \pm 1$.

8. $f(z) \in \mathbb{C}[z]$ gives map $f: \mathbb{C} \rightarrow \mathbb{C}$ $0 \in V$ open nbhd

Let z_1, \dots, z_m be roots of f . Then $\exists z_i \in U_i$, open nbhd, $U_i \cap U_j = \emptyset$ disjoint

$$\begin{aligned} f|_{U_i} &\simeq a(z-z_i)^{m_i} \Rightarrow \deg f|_{z_i} = m_i \\ \Rightarrow \deg(f) &= \sum \deg f|_{z_i} = \sum m_i = \text{degree of } (f) \end{aligned}$$

12. $S^1 \times S^1 \rightarrow S^2 \simeq S^1 \times S^1 / S^1 \vee S^1$

$$\begin{array}{ccccccc} H_2(S^1 \vee S^1) & \rightarrow & H_2(S^1 \times S^1) & \rightarrow & H_2(S^2) & \rightarrow & H_1(S^1 \vee S^1) \rightarrow H_1(S^1 \times S^1) \rightarrow H_1(S^2) \\ \parallel & & \text{iso} & \neq & & & \uparrow \text{iso.} & & \parallel \\ 0 & & & & & & & & 0 \end{array}$$

\Rightarrow Map is not null homotopic

$$\begin{array}{ccc} & & \mathbb{R}^2 \text{ universal cover} \\ & \nearrow g & \downarrow p \\ S^2 & \xrightarrow{f} & S^1 \times S^1 \end{array}$$

Since $\pi_1(S^2) = 0$, \exists lifting $g: S^2 \rightarrow \mathbb{R}^2$ s.t. $pg = f$

$\Rightarrow f_*: H_2(S^2) \rightarrow H_2(S^1 \times S^1)$ factors via $H_2(\mathbb{R}^2) = 0$

Let $g_t: S^2 \rightarrow \mathbb{R}^2$ be a homotopy between g and a constant map, then $pg_t: S^2 \rightarrow S^1 \times S^1$ is a homotopy between f and a constant map.

13. $S^1: \begin{matrix} \bigcirc e \text{ 1-cell} \\ \downarrow v \\ \text{0-cell} \end{matrix}$

$$0 \rightarrow \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 2 \\ 3 \end{pmatrix}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \rightarrow 0$$

X : attach two 2-cells (s_1) degree 2 and 3 (s_2) $\Rightarrow H_2(X) \cong \mathbb{Z} \quad H_1(X) = 0 \quad H_0(X) \cong \mathbb{Z}$.

a)

Subcomplexes. 0) $v \Rightarrow X \cong X/v.$

$$1) S^1 = v \cup e \Rightarrow \begin{matrix} 0 \rightarrow H_2(X) \xrightarrow{\cong \mathbb{Z}} H_2(X/S^1) \rightarrow H_1(S^1) \xrightarrow{\cong \mathbb{Z}} 0 \\ 0 \rightarrow H_1(X/S^1) \rightarrow \tilde{H}_0(S^1) \xrightarrow{\cong} \tilde{H}_0(X) \rightarrow \tilde{H}_0(X/S^1) \rightarrow 0 \end{matrix}$$

$$0 \rightarrow \mathbb{Z}^2 \rightarrow 0 \rightarrow \mathbb{Z} \rightarrow 0 \Rightarrow H_2(X/S^1) \cong \mathbb{Z}^2 \quad H_1(X/S^1) \cong \mathbb{Z}$$

2) $X_2 = v \cup e \cup s_1.$

cellular: $0 \rightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \rightarrow 0$
 $\Rightarrow H_2(X_2) = 0 \quad H_1(X_2) \cong \mathbb{Z}/(2) \quad \tilde{H}_0(X_2) = 0$

Quotient:

$$0 \rightarrow H_2(X_2) \rightarrow H_2(X) \rightarrow H_2(X/X_2) \rightarrow H_1(X_2) \rightarrow H_1(X) \rightarrow H_1(X/X_2) \rightarrow 0$$

$$\begin{matrix} \parallel & \cong & \cong & \cong & \parallel \\ 0 & \mathbb{Z} & \mathbb{Z}/(2) & 0 & 0 \end{matrix}$$

$$\Rightarrow H_2(X/X_2) \cong \mathbb{Z}, \quad H_1(X/X_2) = 0.$$

3) $\mathbb{Z}/3$ in stead of $\mathbb{Z}/2$

$$\begin{aligned} b) \quad X &= (S^1 \cup_2 S_1) \cup_3 S_2 \\ &\cong (S^1 \cup_2 S_1) \cup_1 S_2 \\ &= (S^1 \cup_1 S_2) \cup_2 S_1 \\ &\cong D^2 \cup_2 S_1 \\ &\cong D^2 \cup_0 S_1 \\ &= D^2 \vee S^2 \\ &\cong S^2 \end{aligned}$$

i) In $S^1 \cup_2 S_1$ the one-cell $[e]$ satisfies $[e]^2 = [0]$. Thus $[e]^3 = [e]$

$X \rightarrow X/A$ homotopy equiv.

a) ok.

1) $H_2(X) \not\cong H_2(X/A)$

$$2b) \quad \begin{array}{ccccccc} 0 & \rightarrow & \mathbb{Z}^2 & \xrightarrow{\begin{pmatrix} 2 \\ 3 \end{pmatrix}} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \mathbb{Z} & \xrightarrow{0} & 0 & \xrightarrow{0} & \mathbb{Z} \rightarrow 0 \end{array}$$

$$\begin{array}{ccccccc} \text{ker} & & 1 & \xrightarrow{(3,2)} & \mathbb{Z}^2 & \xrightarrow{\begin{pmatrix} 2 \\ 3 \end{pmatrix}} & \mathbb{Z} \\ 0 & \rightarrow & \mathbb{Z} & \xrightarrow{\quad} & \mathbb{Z}^2 & \xrightarrow{\quad} & \mathbb{Z} \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \mathbb{Z}^3 & \xrightarrow{\quad} & \mathbb{Z} & \xrightarrow{0} & 0 \\ \text{ker} & & & & & & \end{array}$$

$$\Rightarrow H_2(X) \xrightarrow{\cong} H_2(X/A). \quad (\text{no inverses}/\mathbb{Z})$$

$$14. \quad f: S^n \rightarrow S^n \quad f(-x) = f(x) \quad \Leftrightarrow \quad f = f \circ (-1)$$

even map

$$\deg f = \deg f \cdot \deg(-1) = \deg f \cdot (-1)^{n+1}$$

$$\Rightarrow n \text{ even} \Rightarrow \deg(f) = 0.$$

n odd
 f
 even
 map

$$\Rightarrow \begin{array}{ccc} S^n & \xrightarrow{f} & S^n \\ & \searrow p & \nearrow q \\ & \mathbb{R}P^n & \end{array}$$

We have $\mathbb{R}P^{n-1} \hookrightarrow \mathbb{R}P^n \rightarrow \mathbb{R}P^n / \mathbb{R}P^{n-1}$ induces

$$0 \rightarrow H_n(\mathbb{R}P^{n-1}) \rightarrow H_n(\mathbb{R}P^n) \rightarrow H_n(\mathbb{R}P^n / \mathbb{R}P^{n-1}) \rightarrow H_{n-1}(\mathbb{R}P^{n-1})$$

\parallel \uparrow \cong \parallel
 0 iso $H_n(S^n)$ 0
 (by dim.) \downarrow \cong since $n-1$ even
 $(*) \deg(S^n \rightarrow \mathbb{R}P^n) = \deg(S^n \rightarrow \mathbb{R}P^n \rightarrow \mathbb{R}P^n / \mathbb{R}P^{n-1})$

Notice $p^{-1}(\bar{y}) = \{y, -y\}$, thus locally homeomorphism gives

$$\deg p|_y = \deg p|_{-y} = 1$$

* by this we mean $\Rightarrow \deg(p) = 1+1 = 2$

$\deg(gp)$

In general,

$$\mathbb{R}P^n \xrightarrow{g} S^n$$

$$H_n(\mathbb{R}P^n) \xrightarrow{g_*} H_n(S^n)$$

$$\downarrow$$

$$S^n \cong \mathbb{R}P^n / \mathbb{R}P^{n-1}$$

$$\cong \downarrow$$

$$H_n(S^n) \xrightarrow{g_*}$$

$$\Rightarrow \deg g = g_*(1).$$

$$\Rightarrow \deg(f) = 2 \cdot \deg(g).$$

Pick k points in $\mathbb{R}P^n$, x_1, \dots, x_k . Let $x_i \in U_i$ open nbgh, $U_i \cap U_j = \emptyset$

Define $q: \mathbb{R}P^n \rightarrow \mathbb{R}P^n / (\mathbb{R}P^n - \bigcup_i U_i) \cong \bigvee_k S^n$ $q_i: \mathbb{R}P^n \rightarrow S^n$
 projection

Continue $\beta: \bigvee_k S^n \rightarrow S^n$ identifying all k spheres

$$\Rightarrow h = \beta q: \mathbb{R}P^n \rightarrow S^n$$

$$h_*(1) = \sum q_i(1) = k.$$