1. Suppose $f: D^n \to D^n$ fix pt free.

Let $\rho: S^n \to S^n$ reflexion through equatorial plane

Define

$$g: S^n \to S^n; g(x) = \begin{cases} f(\rho(x)) & x \in S^n_+ \approx D^n \\ f(x) & x \in S^n_- \end{cases}$$

$\Rightarrow$ $g$ is continous and fix. pt. free

Define homotopy $F(x,t) = \frac{(1-t)g(x) + t(-1\!\!1)}{\| \quad \|}$ well-defined $\Leftarrow$ fix pt free

$\Rightarrow$ $g \simeq -1\!\!1$ and $\deg(g) = \deg(-1\!\!1) = \pm 1$

But $g$ is not surjective $\Rightarrow$ $\deg(g) = 0$

Contradiction

2.

$f: S^{2n} \to S^{2n}$

a) 1) If $f(x) \neq x$ $\forall x$, then $f \simeq -1\!\!1$ and $\deg(f) = (-1)^{2n-1} = -1$

2) If $f(x) \neq -x$ $\forall x$, then $-f(x) \neq x$ and $\deg(-f) = (-1)^{2n-1} = -1$

$\Rightarrow \deg(f) = 1$

$\Rightarrow$ $f(x) \neq x$ and $f(x) \neq -x$ is impossible

b) We have $S^{2n}/_{x \sim -x} \approx \mathbb{RP}^{2n}$ (universal covering)

$$\begin{array}{ccc} & & S^{2n} \\ & & \downarrow p \\ \mathbb{RP}^{2n} & \xrightarrow{f} & \mathbb{RP}^{2n} \end{array}$$

By lifting criterion $\exists g: \mathbb{RP}^2 \to S^{2n}$ s.t. $pg = f$

Compose with $p$; $pg: S^{2n} \to S^{2n}$. Then $\exists y \in S^{2n}$ s.t.

either $gp(y) = y$ or $gp(y) = -y$. But in either cases

$$f(p(y)) = pgp(y) = p(\pm y) = p(y).$$

$\Rightarrow$ $f$ has a fix pt.

c) $T: \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ without eigenvectors

$T$ induces map $\overline{T}: \mathbb{RP}^{2n-1} \to \mathbb{RP}^{2n-1}$ by

$$\overline{T}([v]) = [T(v)]$$

Then $\overline{T}$ is fix pt free since $\overline{T}([v]) = [v]$ is equiv

to $T(v) = \lambda v$ for some $\lambda \in \mathbb{R}$ $\Rightarrow$ $T$ has eigenvector

3. $f: S^n \to S^n$ $\Rightarrow$ $f$ has a fix pt; $x \in S^n$, $f(x) = x$

$\deg(f) = 0$ But also $-f$ has a fix pt; $y \in S^n$ $-f(y) = y$ or $f(y) = -y$

$F$ vector field on $D^n$ with $F(x) \neq 0 \ \forall x$.

Consider the composition

$$\begin{tikzcd} \partial D^n \arrow[r, "\iota"] & D^n \arrow[r, "\frac{F}{\|F\|}"] & S^{n-1} \\ S^{n-1} \arrow[u, phantom, "\cong"] \arrow[urr, "f"'] & & \end{tikzcd}$$

$$\Rightarrow \quad \begin{tikzcd} \tilde{H}(S^{n-1}) \arrow[r] \arrow[rr, bend right, "0"'] & \tilde{H}(D^n) \arrow[r] & \tilde{H}(S^{n-1}) \end{tikzcd} \qquad \tilde{H}(D^n) = 0$$

$\Rightarrow$ $\deg(f) = 0$ $\Rightarrow$ $\exists\, x \in \partial D^n$ s.t. $f(x) = x$.

But then $F(x) = \|F(x)\| \cdot x$ i.e. points radially outwards

Similar for $f(y) = -y$.

4. Construction of surjective map $f: S^n \to S^n$ of degree 0.

Let $f: S^1 \to S^1$ be given by $f = \mathbb{1}_{S^1} \cdot \overline{\mathbb{1}}_{S^1}$

- surjectiv.
- nullhomotopic $\Rightarrow$ $\deg(f) = \deg(0) = 0$.

But $\deg Sf = \deg f$ and suspension preserves nullhomotopicity.

7. $f: \mathbb{R}^n \to \mathbb{R}^n$ invertible linear transformation

choose basis: $M(f) = A$

Elementary row operations $E$

- Scaling a row; $E_t = t \cdot I + (1-t)E$ positive; $E_t = t(-I) + (1-t)E$ negative } non-sing
- changing two rows (and change sign of one)

$$\begin{tikzcd} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \arrow[d, "\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}"] & & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \arrow[r, "\begin{pmatrix} 1 & 1-t \\ t & 1 \end{pmatrix}"'] & \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \arrow[ur, "\begin{pmatrix} 1-t & -t \\ 1 & 1-t \end{pmatrix}"] & \end{tikzcd}$$

(det $= 1$) $\qquad$ det: $(1-t)^2 + t = 1 - t + t^2$

(det $= 1 - t + t^2 = 0$ $\Rightarrow$ $t = \frac{1 \pm \sqrt{1-4}}{2} \notin [0,1]$)

- Adding multiple of one row

$E_t = t \cdot I + (1-t)E$ non-sing

Composition gives homotopy

$E_t$: $E_0 = A$

$E_1 = \mathrm{diag}(\pm 1)$

$\Rightarrow$ $\det A$ and $\det E_1$ has same sgn.

(all operations preserves sgn of det)

Can assume $A$ is diagonal with entries $\pm 1$.

$$\begin{array}{ccccccc} H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) & \xrightarrow{\approx} & H_{n-1}(\mathbb{R}^n - \{0\}) & \approx & H_{n-1}(S^{n-1}) & \approx & \mathbb{Z} \\ \downarrow A_* & & \downarrow & & & & \downarrow \\ H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) & \xrightarrow{\approx} & H_{n-1}(\mathbb{R}^n - \{0\}) & \approx & H_{n-1}(S^{n-1}) & \approx & \mathbb{Z} \end{array}$$

We have $f = f_1 \cdots f_m$, $f_i$ reflextion through a coordinate hyperpl

$\Rightarrow \deg(f) = (-1)^m = \det(A) = \pm 1$

8. $f(z) \in \mathbb{C}[z]$ gives map $f : \mathbb{C} \to \mathbb{C}$ $0 \in V$ open ngbh

Let $z_1, \dots, z_m$ be roots of $f$. Then $\exists$ $z_i \in U_i$, open ngbh, $U_i \cap U_j = \emptyset$ disjoint

$$f|_{U_i} \approx a(z - z_i)^{m_i} \Rightarrow \deg f|_{z_i} = m_i$$

$$\Rightarrow \deg(f) = \sum \deg f|_{z_i} = \sum m_i = \text{degree of } (f)$$

12. $S^1 \times S^1 \to S^2 \approx S^1 \times S^1 / S^1 \vee S^1$

$$\underset{\substack{\| \\ 0}}{H_2(S^1 \vee S^1)} \to H_2(S^1 \times S^1) \xrightarrow{\text{iso}} H_2(S^2) \to H_1(S^1 \vee S^1) \xrightarrow{\text{iso}} H_1(S^1 \times S^1) \to \underset{\substack{\| \\ 0}}{H_1(S^2)}$$

$\Rightarrow$ Map is not null homotopic

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$$\begin{array}{ccc} & & \mathbb{R}^2 \\ & \overset{\exists g}{\nearrow} & \downarrow p \\ S^2 & \xrightarrow{f} & S^1 \times S^1 \end{array}$$

universal cover

Since $\pi_1(S^2) = 0$, $\exists$ lifting $g : S^2 \to \mathbb{R}^2$ s.t. $pg = f$

$\Rightarrow$ $f_* : H_2(S^2) \to H_2(S^1 \times S^1)$ factors via $H_2(\mathbb{R}^2) = 0$

Let $g_t : S^2 \to \mathbb{R}^2$ be a homotopy between $g$ and a constant map, then $pg_t : S^2 \to S^1 \times S^1$ is a homotopy betwen $f$ and a constant map.

13. $S^1$:  e 1-cell

v

0-cell

$$0 \to \mathbb{Z}^2 \xrightarrow{\binom{2}{3}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

X: attach two 2-cells ($s_1$) degree 2 and 3 ($s_2$)

$$\Rightarrow H_2(X) \approx \mathbb{Z} \quad H_1(X) = 0 \quad H_0(X) \approx \mathbb{Z}.$$

a)

Subcomplexes. 0) $V \Rightarrow X \approx X/v$.

1) $S^1 = v \cup e \Rightarrow$

$$0 \to \underbrace{H_2(X)}_{\approx \mathbb{Z}} \to H_2(X/S^1) \to \underbrace{H_1(S^1)}_{\cong\, \mathbb{Z}} \to 0$$

$$0 \to H_1(X/S^1) \to \tilde{H}_0(S^1) \xrightarrow{\approx} \tilde{H}_0(X) \to \tilde{H}_0(X/S^1) \to 0$$

$$0 \to \mathbb{Z}^2 \to 0 \to \mathbb{Z} \to 0 \qquad \Rightarrow H_2(X/S^1) \approx \mathbb{Z}^2 \quad H_1(X/S^1) \approx \mathbb{Z}.$$

2) $X_2 = V \cup e \cup s_1$.

cellular: $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$

$$\Rightarrow H_2(X_2) = 0 \quad H_1(X_2) \approx \mathbb{Z}/(2) \quad \tilde{H}_0(X_2) = 0$$

Quotient:

$$0 \to \underset{\underset{0}{\parallel}}{H_2(X_2)} \to \underset{\underset{\mathbb{Z}}{\cong}}{H_2(X)} \to H_2(X/X_2) \to \underset{\underset{\mathbb{Z}/(2)}{\cong}}{H_1(X_2)} \to \underset{\underset{0}{\parallel}}{H_1(X)} \to H_1(X/X_2) \to 0$$

$$\Rightarrow H_2(X/X_2) \approx \mathbb{Z}, \quad H_1(X/X_2) = 0.$$

3) $\mathbb{Z}/3$ instead of $\mathbb{Z}/2$

b)

$$\begin{aligned} X &= (S^1 \cup_2 s_1) \cup_3 s_2 \\ &\overset{1)}{\cong} (S^1 \cup_2 s_1) \cup_1 s_2 \\ &= (S^1 \cup_1 s_2) \cup_2 s_1 \\ &\overset{1)}{\cong} D^2 \cup_2 s_1 \\ &\overset{1)}{\simeq} D^2 \cup_0 s_1 \\ &= D^2 \vee S^2 \\ &\simeq S^2 \end{aligned}$$

1) In $S^1 \cup_2 s_1$ the one-cell $[e]$ satisfies $[e]^2 = [0]$. Thus $[e]^3 = [e]$

$X \to X/A$ homotopy equiv.

0) ok.

1) $H_2(X) \not\approx H_2(X/A)$

2,3)

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}^2 \arrow[r, "\binom{2}{3}"] \arrow[d] & \mathbb{Z} \arrow[r, "0"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "0"] & 0 \arrow[r, "0"] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

induces

$$\begin{tikzcd} \text{ker} \arrow[dr] & & & \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "1 \mapsto (3,-2)"] \arrow[d, "3"] & \mathbb{Z}^2 \arrow[r, "\binom{2}{3}"] \arrow[d, "\binom{1}{0}"] & \mathbb{Z} \\ 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & 0 \\ \text{ker} \arrow[ur] & & & \end{tikzcd}$$

$$\Rightarrow H_2(X) \xrightarrow{\cdot 3} H_2(X/A). \quad (\text{no invers}/\mathbb{Z})$$

14. $f: S^n \longrightarrow S^n$ $\quad f(-x) = f(x) \iff f = f \circ (-\mathbb{1})$

<u>even</u> map

$$\deg f = \deg f \cdot \deg(-\mathbb{1}) = \deg f \cdot (-1)^{n+1}$$

$\Rightarrow$ n even $\Rightarrow \deg(f) = 0$.

<u>n odd</u>

f even map $\Rightarrow$

$$\begin{tikzcd} S^n \arrow[rr, "f"] \arrow[dr, "p"'] & & S^n \\ & \mathbb{RP}^n \arrow[ur, "g"'] & \end{tikzcd}$$

We have $\mathbb{RP}^{n-1} \hookrightarrow \mathbb{RP}^n \longrightarrow \mathbb{RP}^n/\mathbb{RP}^{n-1}$ induces

$$\begin{tikzcd} 0 \arrow[r] & H_n(\mathbb{RP}^{n-1}) \arrow[r] \arrow[d, phantom, "\shortparallel"] & H_n(\mathbb{RP}^n) \arrow[r, "\varphi"] & H_n(\mathbb{RP}^n/\mathbb{RP}^{n-1}) \arrow[r] \arrow[d, phantom, "\cong"] & H_{n-1}(\mathbb{RP}^{n-1}) \arrow[d, phantom, "\shortparallel"] \\ & 0 & & H_n(S^n) & 0 \end{tikzcd}$$

(by dim.) $\quad$ $\varphi$ iso $\Downarrow$ $\quad$ since $n-1$ even

(*) $\deg(S^n \to \mathbb{RP}^n)$

$= \deg(S^n \to \mathbb{RP}^n \to \mathbb{RP}^n/\mathbb{RP}^{n-1})$

Notice $p^{-1}(\bar{y}) = \{y, -y\}$, thus locally homeomorphism gives

$$\deg p|_y = \deg p|_{-y} = 1$$

* by this we mean $\deg(gp)$ $\quad \Rightarrow \deg(p) = 1 + 1 = 2$

In general:

$$\begin{tikzcd} \mathbb{RP}^n \arrow[r, "g"] \arrow[d] & S^n \\ S^n \cong \mathbb{RP}^n/\mathbb{RP}^{n-1} & \end{tikzcd} \qquad \begin{tikzcd} H_n(\mathbb{RP}^n) \arrow[r, "g_*"] \arrow[d, "\cong"] & H_n(S^n) \\ H_n(S^n) \arrow[ur, "g_*"'] & \end{tikzcd}$$

$\Rightarrow \deg g = g_*(1)$.

$$\Rightarrow \deg(f) = 2 \cdot \deg(g).$$

Pick $k$ points in $\mathbb{RP}^n$, $x_1, \dots, x_k$. Let $x_i \in U_i$ open ngbh, $U_i \cap U_j = \emptyset$

Define $q: \mathbb{RP}^n \longrightarrow \mathbb{RP}^n/(\mathbb{RP}^n - \bigcup_i U_i) \cong \bigvee_k S^n$ $\quad q_i: \mathbb{RP}^n \to S^n$ projection

Continue $\beta: \bigvee_k S^n \longrightarrow S^n$ identifying all $k$ spheres

$\Rightarrow h = \beta q: \mathbb{RP}^n \longrightarrow S^n$

$$h_*(1) = \sum q_i(1) = k.$$