Repetition:

X
CW-complex

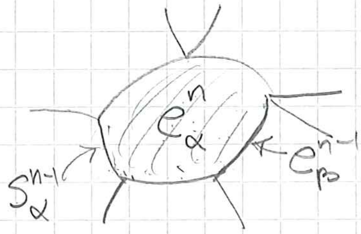
cellular chain complex.

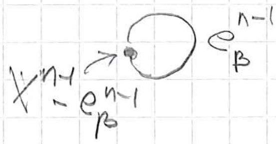
$$\to H_{n+1}(X^{n+1}, X^n) \longrightarrow H_n(X^n, X^{n-1}) \longrightarrow H_{n-1}(X^{n-1}, X^{n-2}) \to$$

cellular boundary formula.

$$d_n(e_\alpha^n) = \sum_\beta d_{\alpha\beta}\, e_\beta^{n-1} \qquad d_{\alpha\beta} = \deg\left(S_\alpha^{n-1} \to X^{n-1} \to S_\beta^{n-1}\right)$$

attaching of e_α^n collapsing $X^{n-1} \setminus e_\beta^{n-1}$ to a point.





Examples.

Euler characteristic

X CW-complex $\qquad \chi(X)$ Euler characteristics $= \sum_n (-1)^n c_n$

c_n: number on n-cells in X

Thm. $\chi(X) = \sum_n (-1)^n \operatorname{rk}(H_n(X))$

Pf.

$$\begin{tikzcd}
 & & \operatorname{im} d_n \arrow[r] & \ker d_{n-1} \arrow[d] & \\
 & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d] & C_n \arrow[r, "d_n"] \arrow[u] & C_{n-1} & \\
0 \arrow[r] & \operatorname{im} d_{n+1} \arrow[r] \arrow[d] & \ker d_n \arrow[r] \arrow[u] & H_n(X) \arrow[r] & 0 \\
 & 0 \arrow[ur] & & &
\end{tikzcd}$$

$-\ \operatorname{rk} C_n = \operatorname{rk} \operatorname{im} d_n + \operatorname{rk} \ker d_n$

$+\ \operatorname{rk} C_{n+1} = \operatorname{rk} \operatorname{im} d_{n+1} + \operatorname{rk} \ker d_{n+1}$

$\operatorname{rk} C_{n-1} = \operatorname{rk} \operatorname{im} d_{n-1} + \operatorname{rk} \ker d_{n-1}$

$$\begin{aligned}
\sum_n (-1)^n \operatorname{rk} C_n &= \sum_n (-1)^n (\operatorname{rk} \operatorname{im} d_n + \operatorname{rk} \ker d_n) \\
&= \sum_n (-1)^n \operatorname{rk} \ker d_n + \sum_n (-1)^n \operatorname{rk} \operatorname{im} d_n \\
&= \sum_n (-1)^n (\operatorname{rk} \ker d_n - \operatorname{rk} \operatorname{im} d_{n+1}) \\
&= \sum_n (-1)^n \operatorname{rk} H_n(X)
\end{aligned}$$

Ex. $\chi(M_g) = 1 - 2g + 1 = 2 - 2g$

$\chi(N_g) = 1 - (g-1) + 1 = 2 - g.$

Suppose: $r : X \xrightarrow{\overset{\iota}{\longleftarrow}} A$, $r\iota = \mathbb{1}$
retraction

$$\Rightarrow \quad \iota_* : H_n(A) \longrightarrow H_n(X) \xrightarrow{r_*} H_n(A)$$

injective.

$$H_n(A) \xrightarrow[\iota_*]{} H_n(X) \longrightarrow H_n(X,A) \xrightarrow[\partial]{} H_{n-1}(A)$$

ker

$$\begin{array}{ccccccc} & C_n(A)/_{im} & \overset{r}{\underset{\longrightarrow}{\longleftarrow}} & C_n(X)/_{im} & \longrightarrow & C_n(X)/_{im} \,/\, C_n(A)/_{im} & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & \ker(A) & \longrightarrow & \ker(X) & \longrightarrow & \ker(X)/\ker(A) & \\ & \downarrow & & & & & \\ & \text{coker.} & & & & & \end{array}$$

$$\Rightarrow \quad 0 \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X,A) \longrightarrow 0$$

short exact.

General knowledge about split shs:

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0 \qquad \text{shs of abelian gps}$$

1) $\exists$ section of i

$\Leftrightarrow$ 2) $\exists$ section of j

3) $B \approx A \oplus B$. (split)

Retraction $r : X \to A \Rightarrow$ split exact.

i.e. $H_n(X) \approx \underline{H_n(A)} \oplus H_n(X,A)$

Example:

Suppose $\exists$ $S^{n-1} \hookrightarrow D^n \xrightarrow[\text{retraction}]{} S^{n-1}$.

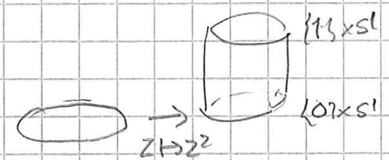
$$0 \to H_k(S^{n-1}) \to H_k(D^n) \to H_k(D^n, S^{n-1}) \to 0 \qquad k>0$$

$H_k(S^{n-1}) \neq 0$ for $k=n-1$, $H_k(D^n) = 0$

Impossible !!.

Example: $f: S^1 \xrightarrow[\text{degree } 2]{} S^1$ $\quad (z \mapsto z^2)$

Mapping cylinder: $(I \times S^1) \sqcup_f S^1 = M_f$

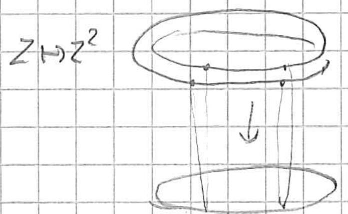


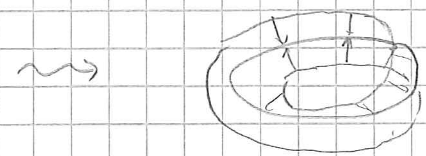
Suppose $S^1 \xrightarrow[\text{inclusion}]{} M_f \xrightarrow[\text{retract}]{} S^1$.

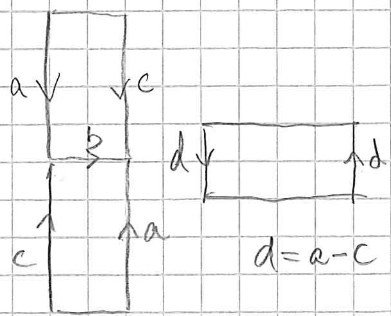
$$0 \to H^1(S^1) \to H^1(M_f) \to H^1(M_f, S^1) \to 0$$

$$0 \to \underset{\cong}{\mathbb{Z}} \xrightarrow{\cdot 2} \underset{\cong}{\mathbb{Z}} \to \mathbb{Z}_2 \to 0$$

But $\mathbb{Z} \not\cong \mathbb{Z} \oplus \mathbb{Z}_2$.







or