Mayer-Vietoris Sequence.

$A, B \subset X \quad , \quad A \cup B = X.$

$$\begin{array}{ccccc} C_n(A+B) := & C_n(A) + C_n(B) & \subseteq & C_n(X) \\ & \downarrow \partial & & \downarrow \partial \\ & C_{n-1}(A) + C_{n-1}(B) & \subseteq & C_{n-1}(X) \end{array}$$

$C_n(A+B) \hookrightarrow C_n(X)$ quasi-isomorphis (induces iso on homology).

$$0 \to C_n(A \cap B) \xrightarrow{\varphi} C_n(A) \oplus C_n(B) \xrightarrow{\psi} C_n(A+B) \to 0$$

$x \longmapsto (x, -x)$

$(x, y) \longmapsto x+y$ exact.

must be checked.

Connecting homomorphism $H_n(X) \xrightarrow{\delta} H_{n-1}(A \cap B)$, $\alpha \in H_n(X)$

α repr. by cycle z, $\alpha = [z]$.

can assume (by barycentric subdiv.) $z = x+y$.

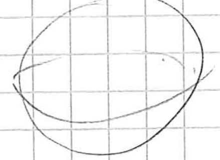
$\partial(x+y) = 0 \Rightarrow \partial x = -\partial y$

$\Rightarrow \delta(\alpha) = [\partial x]$.

$$\Rightarrow \quad \to H_n(A \cap B) \xrightarrow{(1,-1)} H_n(A) \oplus H_n(B) \xrightarrow{+} H_n(X) \xrightarrow{\delta} H_{n-1}(A \cap B)$$

Reduced: OK

Example.



$A \approx B \approx D^n$.

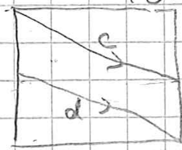
$A \cap B \approx S^{n-1} \times I$

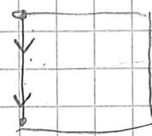
$$\to \tilde{H}_k(S^{n-1}) \to \tilde{H}_k(D^n) \oplus \tilde{H}_k(D^n) \to \tilde{H}_k(S^n) \to \tilde{H}_{k-1}(S^{n-1}) \to .$$

$$\Rightarrow \tilde{H}_k(S^n) \approx \tilde{H}_{k-1}(S^{n-1}).$$

Ex. Klein bottle X $\quad A \quad B \simeq S^1$

Möbiusbands

$A \cap B =$ 

$\simeq$ 

$\dim A\cap B = 1$

$b^{-1} = cd$

$$0 \to H_2(A)\oplus H_2(B) \to H_2(X) \to H_1(A\cap B) \to H_1(A)\oplus H_1(B) \to H_1(X) \to 0$$

$H_2(A) = 0$, $H_2(B) = 0$

$$H_1(A\cap B) \cong \mathbb{Z}(b^{-1}) \to \mathbb{Z}(b_2c)\oplus\mathbb{Z}(a) \cong H_1(A)\oplus H_1(B)$$

$$1 \longmapsto (2,-2)$$

$$b_2c = d\,b_2^{-1} \Rightarrow b^{-1} = cd \mapsto (cd, (cd)^{-1}) \qquad dc^{-1}$$

$$= (b_2^{-1}(b_2c)\,d\,b_2^{-1},\ a^{-1}b_1 b^{-1}a^{-1})$$

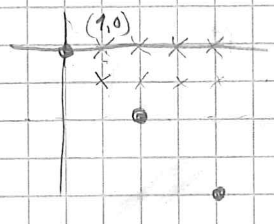
$b_2 \simeq e$ in A

$b_1 \simeq e$ in B

$$= ((b_2c)^2, a^{-2})$$

$$\Rightarrow 1 \longmapsto (2,-2)$$

$$\Rightarrow H_2(X) = 0,\quad H_1(X) \cong \mathbb{Z}^2/(2,-2) \cong \underset{(1,0)}{\mathbb{Z}} \oplus \underset{(1,-1)}{\mathbb{Z}_2}$$

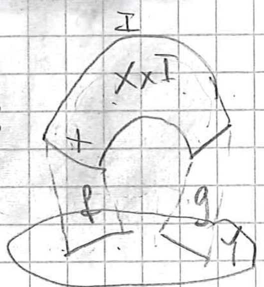


because $2(1,-1) = 0$.

Ex.

$f,g : X \to Y \overset{\iota}{\hookrightarrow} Z$

$(X\times I)\sqcup Y = Z$: $(x,0)\sim f(x)$, $(x,1)\sim g(x)$



$\ker\iota_* = (\alpha,-\alpha) \subset H_n(X)\oplus H_n(X) \cong H_n(X\times\partial I)$, $\quad H_n(X\times I) \cong H_n(X)$ (surjective) (*)

$$\begin{tikzcd}
\overset{*}{0} \arrow[r] & H_{n+1}(X\times I, X\times\partial I) \arrow[r, "\delta"] \arrow[d, "q_*", "\text{iso}"'] & H_n(X\times\partial I) \arrow[r, "\iota_*"] \arrow[d, "q_*"] & H_n(X\times I) \arrow[r] \arrow[d, "q_*"] & 0^{*} \\
\arrow[r] & H_{n+1}(Z,Y) \arrow[r, "\delta"] & H_n(Y) \arrow[r, "\iota_*"] & H_n(Z) \arrow[r] & {}
\end{tikzcd}$$

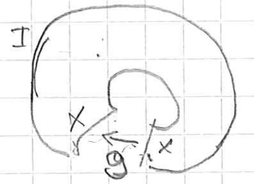
$H_{n+1}(X\times I, X\times\partial I) \cong H_n(X)$, $\quad H_{n+1}(Z,Y) \cong H_n(X)$

$$H_n(Y) \ni f_*(\alpha) - g_*(\alpha) \longmapsto 0$$

$$q : (X\times I, X\times\partial I) \to (Z,Y)$$

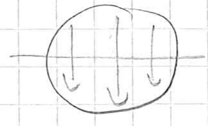
restriction: $((X\times I)\sqcup Y, Y)$ $\xrightarrow{\text{quotient}}$

Suppose $Y = X$, $\varphi = \mathbb{1}$



$(x,0) \sim (g(x),1)$

Ex/ If $X = S^1$ and $g : S^1 \to S^1$ reflection



Z Klein bottle

$X = S^1 = Y$.

$$0 \to H_2(Z) \to H_1(X) \to H_1(Y) \to H_1(Z) \to H_0(X) \to H_0(Y)$$

$$H_2(Z) = 0 \qquad H_1(X) \cong \mathbb{Z} \xrightarrow{2} \mathbb{Z} \cong H_1(Y) \qquad H_0(X) \cong \mathbb{Z} \xrightarrow{0} \mathbb{Z} \cong H_0(Y)$$

$$\alpha \longmapsto \alpha - g_*(\alpha) = 2\alpha \qquad \alpha \longmapsto \alpha - g_*(\alpha) = 0$$

injectvie $\quad g$ reflection

$$\rightsquigarrow \quad 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to H_1(Z) \to \mathbb{Z} \to 0$$

$$\Rightarrow H_1(Z) \approx \mathbb{Z} \oplus \mathbb{Z}/(2)$$