**Proposition**

a) $h: D^k \to S^n$ embedding $\Rightarrow \tilde{H}_i(S^n - h(D^k)) = 0 \ \forall i$

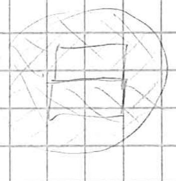
b) $h: S^k \to S^n$ embedding $k < n \Rightarrow \tilde{H}_i(S^n - h(S^k)) = \begin{cases} \mathbb{Z} & i = n-k-1 \\ 0 & \text{else} \end{cases}$

**Pf.** a) By induction on $k$.

$\underline{k=0}$ $\quad S^n - h(D^0) = S^n - pt \approx \mathbb{R}^n$ ok.

Notice $D^k \approx I^k$ (cube)  $\approx$ 

$$= \underset{\substack{\cong \\ I^k}}{I^{k-1} \times [0, \tfrac{1}{2}]} \cup \underset{\substack{\cong \\ I^k}}{I^{k-1} \times [\tfrac{1}{2}, 1]} \qquad A \cap B \approx I^{k-1}$$



$$\underset{\cup}{S^n \setminus h(I^{k-1} \times [0, \tfrac{1}{2}]) \cap S^n \setminus h(I^{k-1} \times [\tfrac{1}{2}, 1])} \approx S^n \setminus h(I^k)$$

$$\approx S^n \setminus h(I^{k-1})$$

Mayer-Vietoris:

$$\underset{\substack{\| \\ 0 \\ \text{by induction}}}{\tilde{H}_{i+1}(S^n \setminus h(I^{k-1}))} \to \tilde{H}_i(S^n \setminus h(I^k)) \to \tilde{H}_i(S^n \setminus h(I^{k-1} \times [0, \tfrac{1}{2}])) \oplus \tilde{H}_i(S^n \setminus h(I^{k-1} \times [\tfrac{1}{2}, 1])) \to$$

$$\tilde{H}_i(S^n - h(I^k)) \neq 0 \Rightarrow \tilde{H}_i(S^n \setminus h(I^{k-1} \times [0, \tfrac{1}{2}])) \neq 0$$

or

$$\tilde{H}_i(S^n - h(I^{k-1} \times [\tfrac{1}{2}, 1])) \neq 0$$

Iteration gives $I = I_0 \supset I_1 \supset I_2 \supset \cdots$ such that

$$\tilde{H}_i(S^n - h(I^{k-1} \times I)) \neq 0 \Rightarrow \tilde{H}_i(S^n - h(I^{k-1} \times I_j)) \neq 0 \ \forall j$$

$$\Rightarrow \tilde{H}_i(S^n \setminus h(I^{k-1} \times \{pt\}))$$

Since $\bigcap I_j = \{pt\}$ we have

$$\tilde{H}_i(S^n \setminus h(I^{k-1} \times \bigcap I_j)) = 0$$

Let $[\alpha] \neq 0$ in $\tilde{H}_i(S^n \setminus h(I^k))$. Then $\alpha = \partial \beta$ ($\beta = \sum n_k \sigma_k$) as singular chain in $S^n \setminus h(I^{k-1} \times \{pt\}) \subseteq \bigcup_j S^n \setminus h(I^{k-1} \times I_j)$

Then by compactness of $\sigma_k$, $\bigcup \operatorname{im} \sigma_k \subset S^n \setminus h(I^{k-1} \times I_j)$ for some $j$

But then $\beta$ is a boundary in $S^n \setminus h(I^{k-1} \times I_j)$ and $[\alpha] = 0$

Contradiction.

b) Induction on $k$

$\underline{k=0}$ $\quad S^n \setminus h(S^0) = S^n - \{p_1, p_2\} \approx \mathbb{R}^n - \{pt\} \simeq S^{n-1} \times \mathbb{R}$

$$\Rightarrow \tilde{H}_i(S^n \setminus h(S^0)) = \tilde{H}_i(S^{n-1} \times \mathbb{R}) = \tilde{H}_i(S^{n-1}) = \begin{cases} \mathbb{Z} & i=n-1 \\ 0 & \text{else} \end{cases}$$

$$S^k = D_+^k \cup D_-^k \quad , \quad D_+^k \cap D_-^k \approx S^{k-1}$$

$$S^n \setminus h(D_+^k) \cap S^n \setminus h(D_-^k) = S^n \setminus h(S^k)$$
$$\cup \qquad\qquad = S^n \setminus h(S^{k-1})$$

Mayer-Vietoris

$$\to \tilde{H}_{i+1}(S^n \setminus h(S^{k-1})) \to \tilde{H}_i(S^n \setminus h(S^k)) \to \underbrace{\tilde{H}_i(S^n \setminus h(D_+^k))}_{=0} \oplus \underbrace{\tilde{H}_i(S^n \setminus h(D_-^k))}_{=0}$$
$$\to \tilde{H}_i(S^n \setminus h(S^{k-1})) \to \dots$$

by a) $\quad \tilde{H}_{i+1}(S^n \setminus h(S^{k-1})) \approx \tilde{H}_i(S^n \setminus h(S^k))$

$= \mathbb{Z}$ for $i+1 = n-(k-1)-1$ or $i = n-k-1$

QED

Consequences:

$n=2$, $k=1$ : $\tilde{H}_i(S^2 - h(S^1)) \approx \mathbb{Z}$ $\quad i = 2-1-1 = 0$

Two components

same for $k=n-1$.

$n=3$ $\quad k=1$ : $\tilde{H}_i(S^3 \setminus h(S^1)) \approx \mathbb{Z}$ $\quad i = 3-1-1 = 1$.

Homology as $S^1$, but fundamental gp. is different.

Theorem

$U \subset \mathbb{R}^n$ open, $h: U \to \mathbb{R}^n$ embedding $\Rightarrow$ $h(U) \subset \mathbb{R}^n$ open.

Pf.

$S^n \approx \mathbb{R}^n \cup \{pos\}$ one-pt-compactification

$\underset{\text{open}}{h(U) \subset \mathbb{R}^n} \iff \underset{\text{open}}{h(U) \subset S^n}$

Let $x \in U$, $\underset{\text{center pt.}}{x \in D^n \subset U}$. Enough to prove $\underset{\text{open}}{h(D^n \setminus \partial D^n) \subset S^n}$

By Prop: $S^n \setminus h(\partial D^n)$ has two path-components $h(D^n \setminus \partial D^n)$ and $S^n \setminus h(D^n)$

1) $h(D^n \setminus \partial D^n) \cap S^n \setminus h(D^n) = \emptyset$

2) $h(D^n \setminus \partial D^n) \approx D^n \setminus \partial D^n$ path-connected

$\tilde{H}_i(S^n \setminus h(D^n)) = 0$ by prop.

$\underset{\text{open}}{S^n \setminus h(\partial D^n) \subset S^n}$ and $\underset{\text{open}}{h(D^n \setminus \partial D^n) \subset S^n \setminus h(\partial D^n)}$

(comp. are open subsets when finitely many)

Corollary.

$M$ compact n-manifold, $N$ connected n-manifold $\Rightarrow$ $h: M \to N$ embedding $\Rightarrow$ homeomorphism

Pf. $h(M) \subset N$ closed by compactness and Hausdorff

$h(M)$ open by theorem

open/closed in connected $\Rightarrow h(M) = N$.

$\varphi : \mathbb{Z}^n \to \mathbb{Z}^n$ $\quad$ $\operatorname{tr}\varphi = \sum a_{ii}$ $\quad$ independent of choice of basis

$(a_{ij})$

General abelian gp $A$ ; $\varphi : A \to A$

$\operatorname{tr}\varphi := \operatorname{tr}\bar{\varphi}$ , $\bar{\varphi} : A/\text{torsion} \to A/\text{torsion}$

$f : X \circlearrowleft$ finite CW-complex

Lefschetz number

$$\tau(f) = \sum_n (-1)^n \operatorname{tr}\left(f_* : H_n(X) \to H_n(X)\right)$$

$\tau(id) = \chi(X)$ Euler characteristic

<u>Thm</u>

$X$ finite simplicial complex $\quad$ $f : X \to X$ s.t. $\tau(f) \neq 0 \Rightarrow f$ has a fix pt.

Pf. Suppose $f$ fix pt-free, By compactness

$\exists$ subdivision $L$ of $X$ and subdv $K$ of $L$

and simplicial map $g : K \to L$ s.t. $g \simeq f$

and $g(\sigma) \cap \sigma = \emptyset$. (exist by the simplicial approximation theorem)

$\Rightarrow \tau(f) = \tau(g)$

$g$ simplicial $\Rightarrow g(K^n) \subset L^n \subset K^n$.

$\Rightarrow g$ induces map $g_* : H_n(K^n, K^{n-1}) \circlearrowleft$

$$\Rightarrow \tau(g) = \sum_n (-1)^n \operatorname{tr}\left(g_*(H_n(K^n, K^{n-1})) \circlearrowleft\right)$$

Trace is additive on exact sequences

$$\begin{array}{ccccccccc} 0 & \to & A & \xrightarrow{\iota} & B & \xrightarrow{\varepsilon} & C & \to & 0 \\ & & \downarrow\alpha & & \downarrow\beta & & \downarrow\gamma & & \\ 0 & \to & A & \xrightarrow{\iota} & B & \xrightarrow{\varepsilon} & C & \to & 0 \end{array} \quad \Rightarrow \operatorname{tr}\beta = \operatorname{tr}\alpha + \operatorname{tr}\gamma$$

$$\beta = \begin{pmatrix} \alpha & * \\ 0 & \gamma \end{pmatrix}$$

$H_n(X)$ is homology of cellular complex

$$\Rightarrow \sum_n (-1)^n \operatorname{tr}(g_* : H_n(X) \circlearrowleft) = \sum_n (-1)^n \underbrace{\operatorname{tr}(g_* : H_n(X^n, X^{n-1}) \circlearrowleft)}_{=0}$$

since $g(\sigma) \cap \sigma = \emptyset$

$$\Rightarrow \tau(g) = \tau(\emptyset) = 0.$$