M : n-manifold : - Hausdorff space

- $x \in U \subset M$ ($U \underset{\text{ss}}{} \mathbb{R}^n$, open)

dim M given intrinsically : $H_i(M, M \setminus \{pt\}) \overset{\text{excision}}{\approx} H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$

$\approx H_{i-1}(\mathbb{R}^n \setminus \{0\})$ $\mathbb{R}^n$ contractible

$\approx H_{i-1}(S^{n-1}) = \mathbb{Z}$ $i = n$.

Compact manifold : Closed.

Orientation : - Preserved under rotation
- Reversed by reflections

Local def : orientation at a point $x \in \mathbb{R}^n$:

choice of a generator of $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \approx \mathbb{Z}$

$\underset{\text{ss}}{} \; H_{n-1}(S^{n-1})$

Rotation has degree 1
Reflection —"— -1

$\rightsquigarrow$ Determines orientation in an other point, since

$$H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \approx H_n(\mathbb{R}^n, \mathbb{R}^n \setminus B) \approx H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{y\})$$

for $x, y \in B$ (ball)

$\rightsquigarrow$ Local orientation : choice of generator for $H_n(M, M \setminus \{x\})$

Orientation : $x \longmapsto \mu_x \in H_n(M, M \setminus \{x\})$

local consistency condition : $x \in B \subset U \subset M$ (B open ball, $U \underset{\text{ss}}{} \mathbb{R}^n$), $\forall y \in B$, $y \neq x$

$$\begin{tikzcd} & \mu_B \in H_n(M, M \setminus B) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ \mu_x \in H_n(M, M \setminus \{x\}) & & \mu_y \in H_n(M, M \setminus \{y\}) \end{tikzcd}$$

If $\exists$ orientation $\Rightarrow$ M is orientable.

Fundamental class:

An element of $H_n(M)$ which maps to generator of $H_n(M, M \setminus \{x\})$ $\forall x \in M$

Thm.

M closed connected n-mfd : $\exists$ fundamental class $\Leftrightarrow$ M is orientable

see 2b

Cap-product:

$$\cap : C_k(X) \times C^\ell(X;\mathbb{Z}) \longrightarrow C_{k-\ell}(X) \qquad k \geq \ell$$

defined by
$$\sigma \cap \varphi = \varphi(\sigma|_{[v_0,..,v_\ell]}) \cdot \sigma|_{[v_\ell, ..., v_k]}$$

Lemma.
$$\partial(\sigma \cap \varphi) = (-1)^\ell(\partial\sigma \cap \varphi - \sigma \cap \delta\varphi)$$

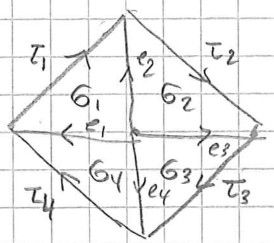
$\rightsquigarrow$ induces
$$H_k(X) \times H^\ell(X;\mathbb{Z}) \xrightarrow{\cap} H_{k-\ell}(X)$$

Functoriality:
$f: X \to Y$

$$\begin{array}{ccccc} H_k(X) & \times & H^\ell(X;\mathbb{Z}) & \longrightarrow & H_{k-\ell}(X) \\ \downarrow f_* & & \uparrow f^* & & \downarrow f_* \\ H_k(Y) & \times & H^\ell(Y;\mathbb{Z}) & \longrightarrow & H_{k-\ell}(Y) \end{array}$$

$$f_*(\alpha) \cap \varphi = f_*(\alpha \cap f^*\varphi)$$

Pf.
$$f\sigma \cap \varphi = \varphi(f\sigma|_{[v_0,..,v_\ell]}) \cdot f\sigma|_{[v_\ell,..,v_k]}$$
$$= \varphi f(\sigma|_{[v_0,..,v_\ell]}) \, f(\sigma|_{[v_\ell,..,v_k]})$$
$$= f_*(\sigma \cap f^*\varphi)$$



$$\alpha = \sum k_i \sigma_i$$

$$\partial\alpha = k_1(\tau_1 - e_2 + e_1) + k_2(\tau_2 - e_3 + e_2) + k_3(\tau_3 - e_4 + e_3) + k_4(\tau_4 - e_1 + e_4)$$

$$= k_1\tau_1 + k_2\tau_2 + k_3\tau_3 + k_4\tau_4 + e_1(k_1 - k_4) + e_2(k_2 - k_1) + e_3(k_3 - k_2) + e_4(k_4 - k_3)$$

$$= 0$$

$\Rightarrow$ $k_1 = k_2 = k_3 = k_4$ and $\tau_1 + \tau_2 + \tau_3 + \tau_4 = 0$

Maps to local generator $k_i = \pm 1$

$\mathbb{RP}^2$: $\tau_1 = \tau_3$, $\tau_2 = \tau_4$ no cycle

S^2 : $\tau_1 = -\tau_4$ $\tau_2 = -\tau_3$ $\exists$ cycle

(torus) T : $\tau_1 = -\tau_3$ $\tau_2 = -\tau_4$ — " — $\Big\}$ orientable.

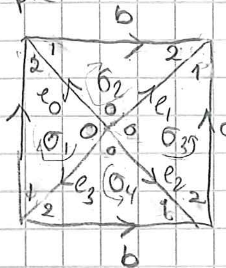
Poincaré duality

M closed orientable n-mfd $\Rightarrow D: H^k(M;\mathbb{Z}) \longrightarrow H_{n-k}(M)$ isomorphism

$\varphi \longmapsto [M] \cap \varphi$

$[M] \in H_n(M)$, fundamental class

Example

T: 

Fundamental class: $[M] = \sigma_1 + \sigma_2 - \sigma_3 - \sigma_4$

Generators for $H^1(T;\mathbb{Z})$: α, β

$\alpha(a) = 1$, $\alpha(e_0) = 1$ $\alpha(e_1) = 1$

$\beta(b) = 1$, $\beta(e_1) = 1$ $\beta(e_2) = 1$

rest $= 0$.

$\Rightarrow [M] \cap \alpha = \sum_i \pm \sigma_i \cap \alpha \quad = b$

$\sigma_1 \cap \alpha = \alpha(\sigma_1|_{[v_0 v_1]}) \cdot \sigma_1|_{[v_1, v_2]} = \alpha(e_3) \cdot (-a) = 0$

$\sigma_2 \cap \alpha = \alpha(\sigma_2|_{[v_0 v_1]}) \cdot \sigma_2[{[v_1, v_2]}] = \alpha(e_0) \cdot (b) = b$

$\sigma_3 \cap \alpha = \alpha(\sigma_3|_{[v_0 v_1]}) \cdot \sigma_3[v_1, v_2] = \alpha(e_2)(a) = 0$

$\sigma_4 \cap \alpha = \alpha(\sigma_4|_{[v_0 v_1]}) \cdot \sigma_4[v_1, v_2] = \alpha(e_3)(b) = 0$

$[M] \cap \beta = a$

Notice. $\psi(\alpha \cap \varphi) = (\varphi \cup \psi)(\alpha)$

Closed, orientable n-mfd $\quad H^k(X;\mathbb{Z}) \times H^{n-k}(X;\mathbb{Z}) \longrightarrow \mathbb{Z}$ non-singular

$(\varphi, \psi) \mapsto (\varphi \cup \psi)(M)$

gives duality between H^k and H_k.

$H^k(X;\mathbb{Z}) \times H_k(X) \to \mathbb{Z}$.

$=$