Lecture Notes on Algebraic Topology for MAT4530 following Allen Hatcher's textbook

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Introduction

0.1 Algebraic Topology

Algebraic topology is largely about problems in algebra and topology, where 'topology' includes the study of topological spaces and continuous maps, and 'algebra' includes the study of groups and homomorphisms.

In this course (MAT4530) we shall study two constructions, leading from topology to algebra, which were essentially introduced by Henri Poincaré in the 1890s. The first, the *fundamental group*, assigns to each based space (X, x_0) a group $\pi_1(X, x_0)$. The second, the *homology groups*, assigns to each space X a sequence of abelian groups $H_n(X)$, one for each $n \geq 0$. By considering all the groups $H_n(X)$ together, rather than for one n at a time, it is possible to axiomatize the properties of the rule taking X to $H_n(X)$. This was done by Eilenberg–Steenrod in the 1940s. During the present course we shall establish as theorems the properties taken as axioms by these authors.

In the sequel to this course (MAT4540), a dual construction to homology, called *cohomology*, is studied. One key property of the sequence of cohomology groups $H^n(X)$ is a bilinear pairing $\cup\colon H^n(X) \times H^m(X) \to H^{n+m}(X)$, which ultimately derives from the diagonal map $X \to X \times X$. The fundamental group will also be put into a more natural context, as the example $i = 1$ of the sequence of *homotopy groups* $\pi_i(X, x_0)$ for $i \geq 1$. The homotopy groups of a space are powerful, but subtle, invariants of that space, and their study forms a core part of *homotopy theory*.

0.2 Lifting Problems

Some geometric questions can be expressed as *lifting problems*, and this leads us to study *covering spaces*.

Consider the unit sphere S^n in Euclidean $(n+1)$-space. For each point $x \in S^n$ the tangent space T_xS^n can be identified with the orthogonal complement $x^\perp = \{t \in \mathbb{R}^{n+1} \mid x \cdot t = 0\}$ of x viewed as a vector in $\mathbb{R}^{n+1}$. These tangent spaces combine to the *total space*

$$TS^n = \{(x, t) \mid x \cdot t = 0\} \subset S^n \times \mathbb{R}^{n+1}$$

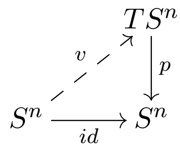
of the *tangent bundle* of S^n. There is a projection map

$$\begin{aligned} p\colon TS^n &\longrightarrow S^n \\ (x, t) &\longmapsto x \end{aligned}$$

with fibers $p^{-1}(x) = T_xS^n$, for all $x \in S^n$. A continuous *vector field* on S^n is a map

$$v\colon S^n \longrightarrow TS^n$$

such that $v(x) \in T_x S^n$, for each $x \in S^n$. Equivalently, the composition $p \circ v$ is the identity map $id\colon S^n \to S^n$.

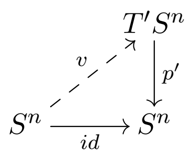


We say that $v\colon S^n \to TS^n$ is a *lifting* of $id\colon S^n \to S^n$ over the projection $p\colon TS^n \to S^n$.

In this case, a lifting always exists, e.g. given by the zero vector field, with $v(x) = 0 \in T_x S^n$, for each $x \in S^n$. A more interesting question is whether there exists a nowhere zero vector field on S^n. To describe this as a lifting problem, we consider the subspace

$$T'S^n = \{(x,t) \mid x \cdot t = 0, t \neq 0\} \subset S^n \times (\mathbb{R}^{n+1} - \{0\})$$

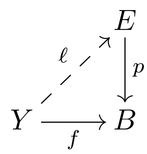
of TS^n given by the nonzero tangent vectors. There is still a projection map $p'\colon T'S^n \to S^n$, and a nowhere zero vector field on S^n is a map $v\colon S^n \to T'S^n$ with $p'v = id$.



A well-known theorem now states that such vectors fields exist for n odd, but not for n even. You cannot 'comb the hair on a coconut' without a cowlick.

A typical proof of this would show that if $p'v = id$ then the identity map $id\colon S^n \to S^n$ can be continuously deformed to the antipodal map $a\colon S^n \to S^n$ (with $a(x) = -x$), and for n even this contradicts the fact that these maps have degree 1 and $(-1)^{n+1} = -1$, respectively. See Theorem 2.28 in [1].

More generally, we can consider a map $p\colon E \to B$ and a space Y. Given a map $f\colon Y \to B$ we may ask if there is a continuous lift $\ell\colon Y \to E$ of f over p, meaning that $p \circ \ell = f$:



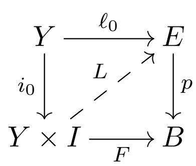
As stated, this question will depend on the choice of map f. However, for many maps $p\colon E \to B$ of spaces the answer does not change if we make a continuous deformation of f, i.e., change f by a *homotopy*.

Recall that a two maps $f_0, f_1\colon Y \to B$ are *homotopic* if there exists a map

$$F\colon Y \times [0,1] \longrightarrow B$$

with $F(y,0) = f_0(y)$ and $F(y,1) = f_1(y)$, for all $y \in Y$. Here $[0,1]$ denotes the unit interval in $\mathbb{R}$, often abbreviated to I. A map F as above is called a *homotopy* from f_0 to f_1, and we write $F\colon f_0 \simeq f_1$.

Definition 0.2.1. A map $p\colon E \to B$ has the *homotopy lifting property* (abbreviated HLP) if for any space Y, any map $\ell_0\colon Y \to E$, and any homotopy $F\colon f_0 \simeq f_1$ of the composite $f_0 = p \circ \ell_0$, there exists a lifted homotopy $L\colon \ell_0 \simeq \ell_1$, with $F = p \circ L$.



Here $i_0 \colon Y \to Y \times I$ denotes the inclusion $i_0(y) = (y, 0)$.

Lemma 0.2.2. *If $p \colon E \to B$ has the homotopy lifting property, $f_0, f_1 \colon Y \to B$ are homotopic maps, and f_0 admits a lift $\ell_0 \colon Y \to E$ over p, then f_1 also admits a lift $\ell_1 \colon Y \to E$ over p.*

Proof. Choose a homotopy $F \colon f_0 \simeq f_1$ and lift it to a homotopy $L \colon \ell_0 \simeq \ell_1$, using the homotopy lifting property. Then ℓ_1 is the required lift of f_1. □

Example 0.2.3. The projection map $p' \colon T'S^n \to S^n$ has the homotopy lifting property. This is an example of a *fiber bundle* (over a numerable base space), all of which have the homotopy lifting property.

When $Y = *$ is a single point, the homotopy lifting property specializes to the path lifting property: For any point $e_0 \in E$ and any path $f \colon I \to B$ from $f(0) = p(e_0)$ there exists a lifted path $\tilde{f} \colon I \to E$ from $\tilde{f}(0) = e_0$ with $p\tilde{f} = f$.

In Section 1.3 of [1] we will discuss *covering spaces*, and prove that they satisfy a strong form of the homotopy lifting property, called the *covering homotopy property* (CHP):

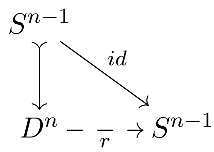
Proposition 0.2.4. *Given a covering space $p \colon E \to B$, a homotopy $F \colon Y \times I \to B$, and a map $\tilde{f} \colon Y \to E$ with $p\tilde{f}(y) = F(y, 0)$ for all $y \in Y$, then there exists a* unique *homotopy $\tilde{F} \colon Y \times I \to E$ with $\tilde{f}(y) = \tilde{F}(y, 0)$ and $p\tilde{F} = F$.*

It will turn out that the fundamental group $\pi_1(X, x_0)$ is an excellent tool for the classification of covering spaces $p \colon \tilde{X} \to X$. For half-reasonable spaces X, the connected based covering spaces $(\tilde{X}, \tilde{x}_0) \to (X, x_0)$ are in one-to-one correspondence with the subgroups of $\pi_1(X, x_0)$. See Theorem 1.38 in [1].

0.3 Extension Problems

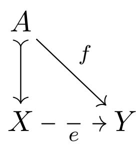
Other geometric questions can be expressed as *extension problems*, and this leads us to study *CW complexes*.

For example, if $f \colon D^n \to D^n$ is any self-map of the unit disc in Euclidean n-space, then Brouwer's fixed point theorem asserts that there exists a point $x \in D^n$ with $f(x) = x$. To prove this, note that if f has no fixed point, then there exists a map $r \colon D^n \to S^{n-1}$, to the boundary $(n-1)$-sphere of that disc, such that $f(x)$, x and $r(x)$ lie on a straight line, in that order. In particular, $r(x) = x$ if $x \in S^{n-1}$, so r is a *retraction* of D^n to the subspace S^{n-1}. In other words, if f has no fixed point, then the identity map $id \colon S^{n-1} \to S^{n-1}$ can be extended over the inclusion $S^{n-1} \subset D^n$.



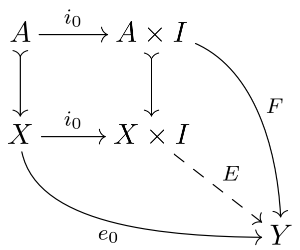
The proof of Brouwer's theorem then proceeds by showing that no such *extension* r of id over $S^{n-1} \subset D^n$ exists.

More generally, one can consider spaces X and Y, and a subspace A of X. Given any map $f \colon A \to Y$ we may ask if there exists an extension $e \colon X \to Y$ of f over $A \subset X$, meaning that e is a map with $e|A = f$:



As stated, this question will depend on the choice of map f. Again, for many pairs (X, A) of spaces the answer does not change if we deform f by a homotopy.

Definition 0.3.1. A pair (X, A) has the *homotopy extension property* (abbreviated HEP) if for any space Y, any map $e_0 \colon X \to Y$, and any homotopy $F \colon f_0 \simeq f_1$ of the restriction $f_0 = e_0|A$, there exists an extended homotopy $E \colon e_0 \simeq e_1$, with $F = E|A \times I$.

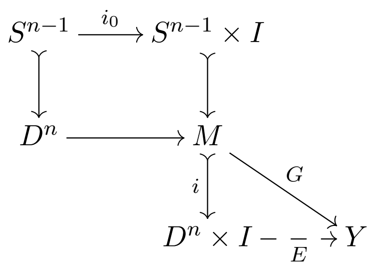


Lemma 0.3.2. *If (X, A) has the homotopy extension property, $f_0, f_1 \colon A \to Y$ are homotopic maps, and f_0 admits an extension $e_0 \colon X \to Y$, then f_1 also admits an extension $e_1 \colon X \to Y$.*

Proof. Choose a homotopy $F \colon f_0 \simeq f_1$ and extend it to a homotopy $E \colon e_0 \simeq e_1$, using the homotopy extension property. Then e_1 is the required extension of f_1. □

Proposition 0.3.3. *The pair (D^n, S^{n-1}) has the homotopy extension property, for each $n \geq 0$.*

Proof. Let $M = D^n \times \{0\} \cup S^{n-1} \times I$ be the *mapping cylinder* of the inclusion $S^{n-1} \subset D^n$. Let $i \colon M \to D^n \times I$ denote the evident inclusion. Given a map $e_0 \colon D^n \to Y$ and a homotopy $F \colon S^{n-1} \times I \to Y$ from $f_0 = e_0|S^{n-1}$ to f_1, we can combine these to a map $G \colon M \to Y$, given by $G(x, 0) = e_0(x)$ for $x \in D^n$ and $G(a, t) = F(a, t)$ for $a \in S^{n-1}$ and $t \in I$.



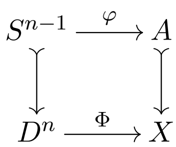
To show that (D^n, S^{n-1}) has the homotopy extension property, we must show that G can be extended over i to a map $E \colon D^n \times I \to Y$. This is achieved by exhibiting a retraction $r \colon D^n \times I \to M$, and setting $E = G \circ r$. Such a retraction r is given by radial projection from $(0, 2) \in D^n \times [0, 2] \subset \mathbb{R}^n \times \mathbb{R}$, so that $(0, 2)$, (x, t) and $r(x, t)$ lie on a straight line, with $(x, t) \in D^n \times I$ and $r(x, t) \in M$. □

We now produce more examples, using (D^n, S^{n-1}) as a building block.

Definition 0.3.4. Let $\varphi \colon S^{n-1} \to A$ be a map. Form the quotient space

$$X = (A \coprod D^n)/\sim ,$$

where $\sim$ denotes the equivalence relation generated by $x \sim \varphi(x)$ for $x \in S^{n-1} \subset D^n$ and $\varphi(x) \in A$. We write $X = A \cup_\varphi D^n$ or $A \cup_\varphi e^n$, and say that we obtain X from A by *attaching an n-cell along φ*.



Note that A is a subspace of X. The canonical map $\Phi\colon D^n \to X$ is called the *characteristic map* of the n-cell. It restricts to a homeomorphism $D^n - S^{n-1} \cong X - A$.

Example 0.3.5. If $A = *$ is a single point, and $\varphi\colon S^{n-1} \to *$ is the only such map, then $X = * \cup_\varphi D^n$ is the quotient space

$$(* \coprod D^n)/\!\sim\; = D^n/S^{n-1}$$

which is homeomorphic to the n-sphere S^n. Hence $S^n \cong * \cup_\varphi e^n$ is obtained by attaching an n-cell to $*$.

We shall see that (X, A) has the homotopy extension property, if X is obtained by attaching a cell to A. We can also attach many cells, in sequence. For simplicity, we start with the 'absolute' case $A = \varnothing$.

Definition 0.3.6. A *CW complex* is a space X with a sequence of subspaces

$$\varnothing = X^{(-1)} \subset X^{(0)} \subset \cdots \subset X^{(n-1)} \subset X^{(n)} \subset \cdots \subset X$$

such that the *n-skeleton* $X^{(n)}$ is obtained from the $(n-1)$-skeleton $X^{(n-1)}$ by attaching n-cells along a set of attaching maps $\{\varphi_\alpha\colon S^{n-1} \to X^{(n-1)}\}_\alpha$, for each $n \ge 0$:

$$\begin{array}{ccc} \coprod_\alpha S^{n-1} & \xrightarrow{\varphi} & X^{(n-1)} \\ \downarrow & & \downarrow \\ \coprod_\alpha D^n & \xrightarrow{\Phi} & X^{(n)} \end{array}$$

Here $\varphi = \coprod_\alpha \varphi_\alpha$ and $\Phi = \coprod_\alpha \Phi_\alpha$, where $\varphi_\alpha\colon S^{n-1} \to X^{(n-1)}$ and $\Phi_\alpha\colon D^n \to X^{(n)}$ are called the attaching map and the characteristic map of the n-cell with index α. In particular, $X^{(0)}$ is any set with the discrete topology. We require that X has the *weak topology* generated by the skeleta, meaning that a subset $L \subset X$ is closed if and only if $L \cap X^{(n)}$ is closed in $X^{(n)}$, for each $n \ge 0$.

The open subsets $U \subset X$ are precisely those for which $U \cap X^{(n)}$ is open in $X^{(n)}$ for all $n \ge 0$. Hatcher simply writes X^n for the n-skeleton. The assumption that all n-cells are attached to the $(n-1)$-skeleton is what distinguishes CW complexes within the larger class of spaces known as *cell complexes*. The prefix 'CW' refers to 'closure finite' and 'weak topology', and originated with a different, but equivalent, definition used by J. H. C. Whitehead, who introduced this class of spaces in the 1940s. For the purposes of algebraic topology, all spaces are 'weakly' equivalent to CW complexes. In particular, for each space Y there is a CW complex X with $\pi_1(X) \cong \pi_1(Y)$ and $H_n(X) \cong H_n(Y)$ for all $n \ge 0$.

Definition 0.3.7. A *relative CW complex* is a pair (X, A) with a sequence of subspaces

$$A = X^{(-1)} \subset X^{(0)} \subset \cdots \subset X^{(n-1)} \subset X^{(n)} \subset \cdots \subset X$$

such that the relative n-skeleton $X^{(n)}$ is obtained from the relative $(n-1)$-skeleton $X^{(n-1)}$ by attaching n-cells along a set of attaching maps $\{\varphi_\alpha\colon S^{n-1} \to X^{(n-1)}\}_\alpha$, for each $n \ge 0$:

$$\begin{array}{ccc} \coprod_\alpha S^{n-1} & \xrightarrow{\varphi} & X^{(n-1)} \\ \downarrow & & \downarrow \\ \coprod_\alpha D^n & \xrightarrow{\Phi} & X^{(n)} \end{array}$$

As in the absolute case, we require that X has the weak topology generated by the relative skeleta $X^{(n)}$.

Proposition 0.3.8. *Any relative CW complex (X, A) has the homotopy extension property.*

Sketch proof. First use Proposition 0.3.3 to deduce that

$$(\coprod_{\alpha \in J} D^n, \coprod_{\alpha \in J} S^{n-1})$$

has the homotopy extension property, for any indexing set J. Then use the definition of $X^{(n)}$ as the union of $X^{(n-1)}$ with $\coprod_\alpha D^n$ along $\coprod_\alpha S^{n-1}$, to deduce that $(X^{(n)}, X^{(n-1)})$ has the homotopy extension property. By induction on $n \geq 0$, it follows that $(X^{(n)}, A)$ has the homotopy extension property, for each $n \geq 0$. Using the assumption that X has the weak topology, it follows that (X, A) has the homotopy extension property. See Proposition 0.16 in [1]. □

Example 0.3.9. The n-sphere S^n admits the structure of a CW complex, with one 0-cell and one n-cell. For $n \geq 1$, the 0-skeleton is a point $*$, and the n-skeleton is all of S^n.

Example 0.3.10. Let *real projective n-space* $\mathbb{R}P^n$ be the quotient space

$$\mathbb{R}P^n = S^n/\sim$$

where $x \sim a(x) = -x$ identifies antipodal points. As a set

$$\mathbb{R}P^n \cong \{L \subset \mathbb{R}^{n+1} \mid \dim L = 1\}$$

can be identified with the set of lines through the origin (1-dimensional real subspaces) of $\mathbb{R}^{n+1}$, taking the equivalence class $[x] = \{x, -x\}$ to the line $L = \mathbb{R}x$, for $x \in S^n \subset \mathbb{R}^{n+1}$.

The inclusions $0 \subset \mathbb{R}^1 \subset \cdots \subset \mathbb{R}^n \subset \mathbb{R}^{n+1}$ then lead to inclusions

$$\varnothing \subset \mathbb{R}P^0 \subset \cdots \subset \mathbb{R}P^{n-1} \subset \mathbb{R}P^n$$

and this sequence is the skeleton filtration of a CW structure on $\mathbb{R}P^n$, with k-skeleton $(\mathbb{R}P^n)^{(k)} = \mathbb{R}P^k$ for each $k \leq n$. Let

$$D^n_+ = \{(x_0, \dots, x_n) \in S^n \mid x_n \geq 0\}$$

be the upper hemisphere in S^n. The projection to the first n coordinates induces a homeomorphism $D^n_+ \cong D^n$, and the composition

$$\Phi_n \colon D^n \cong D^n_+ \subset S^n \xrightarrow{p_n} \mathbb{R}P^n$$

where p_n is the canonical quotient map, is the characteristic map of the single n-cell in $\mathbb{R}P^n$. The attaching map

$$\varphi_n \colon S^{n-1} \longrightarrow \mathbb{R}P^{n-1}$$

is equal to p_{n-1}. Hence

$$\mathbb{R}P^n \cong \mathbb{R}P^{n-1} \cup_{\varphi_n} D^n .$$

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\varphi_n} & \mathbb{R}P^{n-1} \\ \downarrow & & \downarrow \\ D^n & \xrightarrow{\Phi_n} & \mathbb{R}P^n \end{array}$$

More generally, $\mathbb{R}P^n$ has one k-cell for each $0 \leq k \leq n$, with attaching map $\varphi_k = p_{k-1} \colon S^{k-1} \to \mathbb{R}P^{k-1}$:

$$\mathbb{R}P^n = * \cup_{\varphi_1} D^1 \cup_{\varphi_2} \cdots \cup_{\varphi_{n-1}} D^{n-1} \cup_{\varphi_n} D^n .$$

We can also form the infinite-dimensional real projective space

$$\mathbb{R}P^\infty = \bigcup_n \mathbb{R}P^n$$

which is a CW complex with n-skeleton $\mathbb{R}P^n$, having exactly one n-cell for each $n \geq 0$.

Example 0.3.11. Let *complex projective n-space* $\mathbb{C}P^n$ be the quotient space

$$\mathbb{C}P^n = S^{2n+1}/\!\sim$$

where S^{2n+1} is the unit sphere in $\mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$ and $x \sim \lambda x$ for all $\lambda \in S^1 \subset \mathbb{C}$. As a set

$$\mathbb{C}P^n \cong \{L \subset \mathbb{C}^{n+1} \mid \dim L = 1\}$$

can be identified with the set of complex lines through the origin (1-dimensional complex subspaces) of $\mathbb{C}^{n+1}$, taking the equivalence class $[x] = \{\lambda x \mid \lambda \in S^1\}$ to the complex line $L = \mathbb{C}x$, for $x \in S^{2n+1} \subset \mathbb{C}^{n+1}$.

The inclusions $0 \subset \mathbb{C}^1 \subset \cdots \subset \mathbb{C}^n \subset \mathbb{C}^{n+1}$ then lead to inclusions

$$\varnothing \subset \mathbb{C}P^0 \subset \cdots \subset \mathbb{C}P^{n-1} \subset \mathbb{C}P^n$$

and this sequence is the skeleton filtration of a CW structure on $\mathbb{C}P^n$, with $2k$-skeleton $(\mathbb{C}P^n)^{(2k)} = \mathbb{C}P^k$ for each $k \le n$. Let

$$D^{2n}_+ = \{(x_0, \dots, x_n) \in S^{2n+1} \mid x_n \ge 0\}$$

be the subspace of $S^{2n+1} \subset \mathbb{C}^{n+1}$ where the last complex coordinate, $x_n \in \mathbb{C}$, is real and non-negative. Note that

$$x_n = \sqrt{1 - |x_0|^2 - \cdots - |x_{n-1}|^2}$$

for $(x_0, \dots, x_n) \in D^{2n}_+$. The projection to the first n complex coordinates induces a homeomorphism $D^{2n}_+ \cong D^{2n} \subset \mathbb{C}^n$, and the composition

$$\Phi_{2n} \colon D^{2n} \cong D^{2n}_+ \subset S^{2n+1} \xrightarrow{q_n} \mathbb{C}P^n$$

where q_n is the canonical quotient map, is the characteristic map of the single $2n$-cell in $\mathbb{C}P^n$. The attaching map

$$\varphi_{2n} \colon S^{2n-1} \longrightarrow \mathbb{C}P^{n-1}$$

is equal to q_{n-1}. Hence

$$\mathbb{C}P^n \cong \mathbb{C}P^{n-1} \cup_{\varphi_{2n}} D^{2n} .$$

$$\begin{array}{ccc} S^{2n-1} & \xrightarrow{\varphi_{2n}} & \mathbb{C}P^{n-1} \\ \downarrow & & \downarrow \\ D^{2n} & \xrightarrow{\Phi_{2n}} & \mathbb{C}P^n \end{array}$$

More generally, $\mathbb{C}P^n$ has one $2k$-cell for each $0 \le k \le n$, with attaching map $\varphi_{2k} = q_{k-1} \colon S^{2k-1} \to \mathbb{C}P^{k-1}$:

$$\mathbb{C}P^n = * \cup_{\varphi_2} D^2 \cup_{\varphi_4} \cdots \cup_{\varphi_{2n-2}} D^{2n-2} \cup_{\varphi_{2n}} D^{2n} .$$

We can also form the infinite-dimensional complex projective space

$$\mathbb{C}P^\infty = \bigcup_n \mathbb{C}P^n$$

which is a CW complex with $2n$-skeleton $\mathbb{C}P^n$, having exactly one $2n$-cell for each $n \ge 0$, and no odd-dimensional cells.

The maps $p_n\colon S^n \to \mathbb{R}P^n$ are examples of covering maps. The maps $q_n\colon S^{2n+1} \to \mathbb{C}P^n$ are not covering maps, but they have the homotopy lifting property, and are examples of 'fiber bundles', which also include the projection map $p'\colon T'S^n \to S^n$ previously discussed.

There is a close connection between the cell structure of a CW complex X and the homology groups $H_n(X)$ of that space. The attaching maps of the $(n+1)$- and n-cells determine two homomorphisms

$$\bigoplus_\alpha \mathbb{Z} \xrightarrow{\partial_{n+1}} \bigoplus_\beta \mathbb{Z} \xrightarrow{\partial_n} \bigoplus_\gamma \mathbb{Z}$$

where α, β and γ run over the sets of $(n+1)$-cells, n-cells and $(n-1)$-cells in X, respectively. The group $C_n(X) = \bigoplus_\beta \mathbb{Z}$ is called the group of cellular n-chains in X. The image $B_n(X) = \operatorname{im}(\partial_{n+1})$ is called the group of n-boundaries, and the kernel $Z_n(X) = \ker(\partial_n)$ is called the group of n-cycles. The composite $\partial_n \circ \partial_{n+1}$ is zero, so

$$B_n(X) \subset Z_n(X) \subset C_n(X)\,.$$

To measure whether each n-cycle is an n-boundary we consider the quotient group

$$H_n(X) = Z_n(X)/B_n(X)\,.$$

This is the n-th (cellular) homology group of X. It measures, in some sense, the presence of n-dimensional 'holes' in X. See Section 2.2 in [1].

Example 0.3.12. $H_{2k}(\mathbb{C}P^n) \cong \mathbb{Z}$ for each $0 \le k \le n$, while the remaining groups $H_i(\mathbb{C}P^n)$ are trivial.

0.4 Homotopy Functors

Lifting problems for maps to $p\colon E \to B$, where p has the homotopy lifting property, only depend on the homotopy class $[f]$ of the map $f\colon Y \to B$ that is to be lifted. In the same way, extension problems for maps out of $i\colon A \to X$, where i has the homotopy extension property, only depend on the homotopy class $[f]$ of the map $f\colon A \to Y$ to be extended. It is therefore possible to address these questions using constructions that are insensitive to deformations through continuous maps, i.e., homotopies.

The constructions π_1 and H_n are examples of such constructions, or *functors*, from spaces to groups, taking maps to homomorphisms. The condition of functoriality implies that these constructions take homeomorphisms to isomorphisms, so that if $X \cong Y$ then $H_n(X) \cong H_n(Y)$ for all $n \ge 0$, and similarly for π_1 in the based case.

Moreover, π_1 and H_n are homotopy invariant, in the sense that homotopic maps are taken to the same homomorphism. For instance, if $f \simeq f'\colon X \to Y$, then $H_n(f) = H_n(f')\colon H_n(X) \to H_n(Y)$ for each $n \ge 0$. This implies that not only homeomorphic spaces X and Y have isomorphic homology groups, but also any pair of *homotopy equivalent* spaces:

Definition 0.4.1. A map $f\colon X \to Y$ is a *homotopy equivalence* if there exists a map $g\colon Y \to X$, called the *homotopy inverse*, such that there are homotopies $gf \simeq id\colon X \to X$ and $fg \simeq id\colon Y \to Y$. In this case we say that X and Y are *homotopy equivalent*, and write $X \simeq Y$.

There is therefore an intermediate context, or *category*, between topology and algebra, called the *homotopy category*. The objects of study are still topological spaces, but the *morphisms* from a space X to a space Y are not the set of maps $f\colon X \to Y$, but instead the quotient set of homotopy classes $[f]$ of such a map.

In this category, the homotopy classes $[f]$ and $[g]$ of a homotopy equivalence f and its homotopy inverse g are strict inverses, in the sense that $[g][f] = id_X$ and $[f][g] = id_Y$ are equal, as homotopy classes of maps.

Questions in topology can therefore be studied in two steps, first by comparison with a homotopy invariant problem in the homotopy category, and then by means of π_1 and the H_n in terms of a problem in algebra.

$$
\begin{array}{ccccc}
\text{Topological Spaces} & \longrightarrow & \text{Homotopy Category} & \longrightarrow & \text{Graded Abelian Groups} \\
X & \longmapsto & X & \longmapsto & H_*(X) \\
f & \longmapsto & [f] & \longmapsto & f_*
\end{array}
$$

Taking the cup product into account, the second map can be further refined as a map to 'Graded Rings' (using cohomology). Alteratively, it can be extended through the 'Stable Homotopy Category', which is even closer to algebra than the ordinary homotopy category.

Bibliography

[1] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR1867354