Solution problem 0.18

Recall the definition of the join $X * Y = X \times Y \times I /_\sim$

where $(x, y_1, 0) \sim (x, y_2, 0)$ $\forall x \in X$, $\forall y_1, y_2 \in Y$

and $(x_1, y, 0) \sim (x_2, y, 0)$ $\forall x_1, x_2 \in X$ and $\forall y \in Y$.

Let us define

$$S^1 \times S^1 \times I \xrightarrow{f} S^3 \quad \text{by}$$

$$f(x, y, t) = \frac{((1-t)x, ty)}{\sqrt{(1-t)^2 + t^2}} \in S^3$$

~~Assume~~ $f(x_1, y_1, t_1) = f(x_2, y_2, t_2)$

Then $$\frac{(1-t_1)x_1}{\sqrt{(1-t_1)^2 + t_1^2}} = \frac{(1-t_2)x_2}{\sqrt{(1-t_2)^2 + t_2^2}} \quad \text{and}$$

$$\frac{t_1 y_1}{\sqrt{(1-t_1)^2 + t_1^2}} = \frac{t_2 y_2}{\sqrt{(1-t_2)^2 + t_2^2}}.$$

If $t_1, t_2 \notin \{0, 1\}$, we must have $x_1 = x_2$ and $y_1 = y_2$ (since say $x_1 \in \mathbb{R}^2$ must be a positive multiple of x_2 and $x_1, x_2 \in S^1$) and $t_1 = t_2$ since $t \to \frac{t}{\sqrt{(1-t)^2 + t^2}}$ is increasing on $[0, 1]$.

If $t_1 = 1$, then $\frac{(1-t_1)x_1}{\sqrt{(1-t_1)^2 + \cancel{t_1}^2}} = \vec{0} = \frac{(1-t_2)\vec{x_2}}{\sqrt{(1-t_2)^2 + \cancel{t_2}^2}}$

hence $t_2 = 1$. So we get ~~$\vec{y_1} = \vec{y_2}$~~. $y_1 = y_2$.

Likewize $t_1 = 0$ implies $t_2 = 0$ and ~~$x_1 = x_2$~~. $x_1 = x_2$.

This shows that $f(x_1, y_1, t_1) = f(x_2, y_2, t_2) \Leftrightarrow (x_1, y_1, t_1) \sim (x_2, y_2, t_2)$

and therefore that $S^1 \times S^1 = S^1 \times S^1 \times I / f$.

f will therefore induce and injective map

$\bar{f}: S^1 \times S^1 \to S^3$.

Let $\vec{z} = (z_1, z_2) \in S^3$ with $z_1, z_2 \in \mathbb{R}^2$

Let us first assume that $z_1 \neq \vec{0}$ and $z_2 \neq \vec{0}$

Let $t = \dfrac{|z_2|}{|z_1| + |z_2|}$ hence $1 - t = \dfrac{|z_1|}{|z_1| + |z_2|}$. Let $x = \dfrac{z_1}{|z_1|}$ and $y = \dfrac{z_2}{|z_2|}$

a direct calculation show that

$f(x, y, t) = (z_1, z_2)$. ~~hence~~

Let now $z_1 = \vec{0}$, Let x be arbitrary in S^1.

Now $\|z_2\| = 1$ so let $y = z_2 \in S^1$.

then $f(x, y, 1) = (z_1, z_2) = (0, z_2)$

If $\vec{z}_2 = \vec{0}$, we choose $t = 0$, $x = z_1$, $y \in S^1$

arbitrary and $f(x, y, 0) = (z_1, z_2) = (z_1, 0)$

This shows that f hence $\bar{f}$ is surjective.

Now $\bar{f}$ is a bijective map from a compact space to a Hausdorff space. Such a map is closed, hence a homeomorphism. The proof for $S^m \times S^n \cong S^{m+n+1}$ is similar using the same formulas for f writing $\vec{z} = (z_1, z_2)$ with $z_1 \in \mathbb{R}^{m+1}$, $z_2 \in \mathbb{R}^{n+1}$.