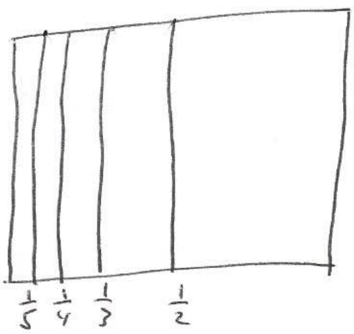
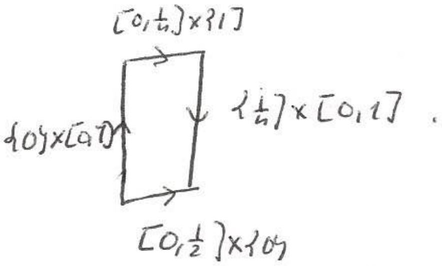
Exercise 1.3.4

Let X be the space consisting of the four sides of $[0,1]\times[0,1]$, together with $\bigcup_{n=2}^{\infty}\{\frac{1}{n}\}\times[0,1]$.

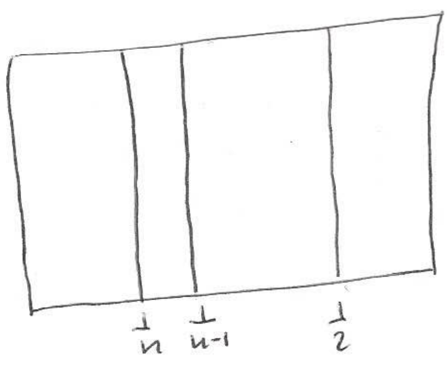


X. Assume $p:\tilde{X}\to X$ is a covering space

Let C_n be the loop



We will see that C_n is not homotopically trivial in X. Let Y_n be the subspace of X we get by deleting all points (x,y) such that $x\in(0,\frac{1}{n})$ and $y\in(0,1)$.



Y_n

By collapsing the contractible subcomplex $\{0\}\times[0,1]\cup[0,1]\times\{0,1\}$ we see that Y_n is homotopy-equivalent with the wedge of n-circles (Prop 0.17). Now (ex. 1.2.1) $\pi_1(Y_n)$ is the free product of n-copies of $\mathbb{Z}$ and $[C_n]$ is one of the generators.

We have a map $X \xrightarrow{r} Y_n$ ~~by~~ mapping all points (x,y) with $x \in [0, \frac{1}{n+1}]$ to $(0,y)$ such that $r(C_n) = C_n$. So C_n cannot be homotopically trivial in X.

Now consider $Z = \{0\} \times [0,1]$. Since Z is compact we may find neighbourhoods $U_i, i=0,..,m$ such that $Z \in \bigcup_{i=0}^{m} U_i$, $U_i \cap U_{i+1} \neq \emptyset$ and $U_i \cap U_j = \emptyset$ if $|i-j|>1$.

We assume each U_i is evenly covered by p. So we may lift each U_i to $\tilde{X}$. Assume that we have lifted $\bigcup_{i=0}^{k-1} U_i$. Since $U_{k-1} \cap U_k$ also is an evenly covered neighbourhood there is a unique way to lift further to U_k such that the lifte agrees to the one we had on $U_{k-1} \cap U_k$.

So finally we get a lift $\tilde{f}: \bigcup_{i=0}^{m} U_i \to \tilde{X}$. Put $U = \bigcup_{i=0}^{m} U_i$

Now let $X_n = \{(x,y) \in X \mid x \in [0, \frac{1}{n}]\}$. For n big we have $X_n \subset U$, Now X_n is also a neighbourhood of Z. $\tilde{f}: X_n \to \tilde{X}$ is a continuous injective map from a compact space to a Hausdorff space, ($\tilde{X}$ is Hausdorff since X is Hausdorff), so $\tilde{f}$ is a homeomorphism on its image. Let $\tilde{X}_n = \tilde{f}(X_n)$

Let $\tilde{C}_n = f(C_n)$, Since $[C_n] = [p\tilde{C}_n] \neq [0]$ we cannot have $[\tilde{C}_n] = [0]$, so $\tilde{X}$ cannot be simply connected.

Exercise 1.3.9

Let X be a space $f: X \to S^1$ a map.

Assume that $\pi_1(X)$ is finite.

We will show that X is null-homotopic.

Let $g \in \pi_1(X)$, $\cancel{nf_*}\, f_*(g) = m \in \mathbb{Z} = \pi_1(S^1)$. Assume $g \neq 0$

Now since $\pi_1(X)$ is finite $\exists\, n > 1$ such that

$g^n = 0$, hence $f_*(g^n) = nm = 0$, so $m = 0$

and $f_*(\pi_1(X)) = \{0\}$.

We have assumed that X is path-connected and locally path connected. Let $p: \mathbb{R} \to S^1$ be the ~~univer~~ simply-connected covering space of S^1. Since $f_*(\pi_1(X)) = \{0\} = p_*(\pi_1(\mathbb{R}))$, f has a lift

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & S^1 \end{tikzcd}$$

Since $\mathbb{R}$ is contractible $\tilde{f}$ is null homotopic, hence $f = p\tilde{f}$ is also null-homotopic.

1.3.25

$\varphi(x,y) = (2x, y/2)$. Given $(x,y) \neq (0,0)$, we define an action of $\mathbb{Z}$ by $n \cdot (x,y) = \varphi^n(x,y) = (2^n x, y/2^n)$

We must see that this action "properly discontinuous" that is for each $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$, $\exists$ a neighbourhd U

Such that $\varphi^n(U) \cap \varphi^m(U) = \emptyset$ when $n \neq m$.

(here if $n=-m$, $m>0$ $\varphi^n(x,y) = (2^{-m}x, 2^m y)$.

Now let $x \neq 0$, $x>0$, let $0<\varepsilon<\frac{x}{3}$. Put $s_0 = x-\varepsilon$

$t_0 = x+\varepsilon$, and $s_n = 2^n s_0$, $t_n = 2^n t_0$.

It is clear that $s_0 < s_1 < \dots < s_n < s_{n+1}$ and $t_0 < t_1 < \quad < t_n < t_{n+1}$. So the intervals (s_n, t_n) $n \geq 0$ becomes disjoint if $s_{n+1} > t_n$.

So then $2^{n+1} s_0 > 2^n t_0 \Leftrightarrow 2s_0 > t_0 \Leftrightarrow \varepsilon < \frac{x}{3}$.

Considering $s_0 > s_{-1} > s_{-2} > \quad > s_{-n} >$ and $t_0 > t_{-1} > t_{-2} > \quad > t_{-n} > \dots$ the intervals (s_n, t_n) $n \leq 0$ becomes disjoint if also if $s_{n+1} > t_n \Leftrightarrow \varepsilon < \frac{x}{3}$. (Here $0<\varepsilon<\frac{|x|}{3}$)

If $x<0$ the arguments are similar. Also if $y \neq 0$ and $0<\varepsilon<\frac{|y|}{3}$, we se that the intervals $(2^n(y-\varepsilon), 2^n(y+\varepsilon))$ becomes disjoint ~~for~~ when $n \in \mathbb{Z}$. by the same argument.

~~It follows that the~~

Now if $(x,y) \neq \bar{0}$ and $x \neq 0$, we se that $(x-\varepsilon, x+\varepsilon) \times I$ (I any interval) $0<\varepsilon<|x|/3$, ~~have are dis~~ will satisfy (*) on p. 72.

the case $i_2 \neq 0$ is similar.

To calculate $\pi_1(X/\mathbb{Z})$ we have an exact sequence

$$0 \to \pi_1(X) \xrightarrow{p} \pi_1(X/\mathbb{Z}) \to \pi_1(X/\mathbb{Z}) \Big/ p_*(\pi_1(X/\mathbb{Z})) \to 0$$

By 1.40 (c) $\pi_1(X/\mathbb{Z}) / p_*(\pi_1(X/\mathbb{Z})) \approx \mathbb{Z}$

also $\pi_1(X) = \pi_1(\mathbb{R}^2 - \{(0,0)\}) \approx \pi_1(S^1) \approx \mathbb{Z}$

So we have an exact sequence

$$0 \to \mathbb{Z} \to \pi_1(X/\mathbb{Z}) \to \mathbb{Z} \to 0$$

Since $\mathbb{Z}$ is a free group this is a so-called split exact sequence hence $\pi_1(X/\mathbb{Z}) \approx \mathbb{Z} \oplus \mathbb{Z}$.

(see p. 147 and 148).

To see that $X/\mathbb{Z}$ is not Hausdorff:

Consider $(1,0)$ and $(0,1)$ in $\mathbb{R}^2 - \{0\}$.

Let $(1-\varepsilon, 1+\varepsilon) \times (-\delta, \delta)$ and $(-\eta, \eta) \times (1-\xi, 1+\xi)$ be nbd of $(1,0)$ and $(0,1)$

~~Let $p: X \to X/\mathbb{Z}$, and let $\bar{U}_1 = p((1-\varepsilon$~~

~~Put $U_1 =$~~

Denote these neighbourhood U_1, U_2

Let $p: X \to X/\mathbb{Z}$, $\bar{U}_1 = p(U_1)$, $\bar{U}_2 = p(U_2)$

Any nbds of $p(1,0), p(0,1)$ will contain nbd's of type $\bar{U}_1, \bar{U}_2$.

Now $\varphi^{-n}(\mathcal{U}_1) = (2^{-n}(1-\varepsilon), 2^{-n}(1+\varepsilon)) \times (-2^n\delta, 2^n\delta)$

and $\varphi^n(\mathcal{U}_2) = (-2^n\eta, 2^n\eta) \times (2^{-n}(1-\xi), 2^{-n}(1+\xi))$

it is clear that when n is large $\varphi^{-n}(\mathcal{U}_1) \cap \varphi^n(\mathcal{U}_2) \neq \phi$

So $\overline{\mathcal{U}}_1 \cap \overline{\mathcal{U}}_2 = p(\varphi^{-n}(\mathcal{U}_1)) \cap p(\varphi^n(\mathcal{U}_2)) \neq \phi$.

So $X/\mathbb{Z}$ is not Hausdorff.

Consider

$\{(x,y) \mid x>0, y>0\}$. Consider a hyperbola

$\{xy = c\}$ Now $2^n x \cdot 2^{-n} y = c$ so $\{xy = c\}$

is invariant under the action of $\mathbb{Z}$.

Also $\{xy = c\}/\mathbb{Z}$ is a circle since points

$(x,y) \underset{\mathbb{Z}}{\sim} (2^n x, 2^{-n} y)$, $n \in \mathbb{Z}$ so this is similar to

the action of $\mathbb{Z}$ in $\mathbb{R}$ by translation $(x,y) \to (x+n, y+n)$

Now let us denote $\{xy = c\}$ by H_c.

Then $\{(x,y) \mid x>0, y>0\} = \bigcup_{c>0} H_c$ and since $H_c/\mathbb{Z} \approx S^1$

we will get $\{(x,y) \mid x>0, y>0\}/\mathbb{Z} \approx S^1 \times \mathbb{R}$