Exact sequences

$\{A_n \mid n \in \mathbb{Z}\}$ a collection of abelian groups.

$A_{n+1} \xrightarrow{\alpha_{n+1}} A_n$ a homeomorphism for each n

Def
The sequence

$$\to A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \to$$

is exact if $\operatorname{im}\alpha_{n+1} = \ker \alpha_n$ for each n.

Examples

i) $0 \to A \xrightarrow{\alpha} B$ is exact $\Leftrightarrow$ is injective

ii) $A \xrightarrow{\alpha} B \to 0$ is exact $\Leftrightarrow$ α is surjetive

iii) $0 \to A \xrightarrow{\alpha} B \to 0$ is exact $\Leftrightarrow$ α is an isomorphism

(iv) $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$

is exact $\Leftrightarrow$ α is injective, β is surjective and $\operatorname{im}\alpha = \ker\beta$

(iv) is called a short exact sequence.

If C is a free group it can be proved that if iv) is exact then $B \simeq A \oplus C$

Let X be a space $A \subset X$.
(X, A) is called a „good" pair if $\exists$ a neighbourhood U of A such that A is a deformation retract of U
(That is $\exists$ $r_t : U \to U$ such that $r_t | A = 1\!\!1_A \; \forall t$
$r_0 = 1\!\!1_U$ and $r_1(U) = A$)

Now we have maps $i : A \to X$ and $j : X \to X/A$, and induced map i_*, j_* in homology.

<u>Theorem 2.13</u>
There ~~exists~~ is an exact sequence

$$\xrightarrow{\partial} \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{j_*} \tilde{H}_n(X/A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \xrightarrow{i_*}$$

(Here we assume that (X, A) is „good")

<u>Proof</u> (later)

We will look at some applications.

Example of „good" pairs. X a CW-complex A a subcomplex (see the appendix for a proof).

Example (Cor. 2.14)

$\tilde{H}_n(S^n) \approx \mathbb{Z}$ and $\tilde{H}_i(S^n) = 0$ for $i \neq n$.

<u>Proof</u>

When $n = 0$, $S^0 = \{-1, 1\}$. $H_0(S^0) \approx \mathbb{Z} \oplus \mathbb{Z}$

and $H_0(S^0) \approx \tilde{H}_0(S^0) \oplus \mathbb{Z}$ hence $\tilde{H}(S^0) \approx \mathbb{Z}$

Also $\tilde{H}_i(S^0) \approx H_i(S^0) \approx H_i(\{-1\}) \oplus H_i(\{1\}) = 0$ $i \neq 0$.

Assume that the stament is true for all S^k with $0 \leq k < n$.

Let $(X, A) = (D^n, S^{n-1})$ this a good pair (since D^n is a CW-complex S^{n-1} a subcomplex)

Now $D^n/S^{n-1} \approx S^n$

and

$$\to \tilde{H}_i(D^n) \to \tilde{H}_i(D^n/S^{n-1}) \approx \tilde{H}_i(S^n) \xrightarrow{\partial} \tilde{H}_{i-1}(S^{n-1}) \to \tilde{H}_{i-1}(D^n) \to$$

is exact. D^n is contractible, so $\tilde{H}_i(D^n) = 0 \ \forall i$

so $\tilde{H}_i(S^n) \approx \tilde{H}_{i-1}(S^{n-1}) \quad \forall i$.

The statement follows by induction.

Example (Cor. 2.15)

$S^{n-1} = \partial D^n$ is not ~~an~~ retract of D^n.

Proof

If $r : D^n \to S^{n-1}$ is a retract, then
$1\!\!1_{S^{n-1}} = r \circ i$ when $i : S^{n-1} \to D^n$ is inclusion.

Then we get

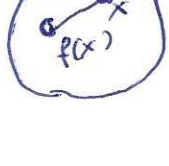
$$\mathbb{Z} \cong \tilde{H}_{n-1}(S^{n-1}) \xrightarrow{i_*} \tilde{H}_{n-1}(D^n) = 0 \xrightarrow{r_*} \tilde{H}_{n-1}(S^{n-1}) \cong \mathbb{Z}$$

So we get

$$\mathbb{Z} \xrightarrow{i_*} 0 \xrightarrow{r_*} \mathbb{Z} \quad \text{and } r_* \circ i_* = 1\!\!1_{\mathbb{Z}}$$

impossible.

This implies:
Every map $f : D^n \to D^n$ has a fix-point.
If not $f(x) \neq x \ \forall x \in D^n$. Let R be the ray from $f(x)$ going through x.
R intersect $\partial D^n = S^{n-1}$ in exactly one point $r(x)$. and $r : D^n \to S^{n-1}$ becomes a retraction – Impossible.



$r(x)$. ($r(x) = x$ if $x \in S^{n-1}$)

Relative homology groups

$A \subset X$, $C_n(A)$ is a subgroup of $C_n(X)$

Let $C_n(X,A) = C_n(X)/C_n(A)$.

Since $\partial : C_n(X) \to C_{n-1}(X)$ restrict to $C_n(A)$ and $\partial C_n(A) \subset C_{n-1}(A)$ we may define a boundary map

$$\partial : C_n(X,A) \to C_{n-1}(X,A)$$

We get a chain complex

$$\cdots \to C_{n+1}(X,A) \xrightarrow{\partial} C_n(X,A) \xrightarrow{\partial} C_{n-1}(X,A) \to$$

and may define homology of this chain complex $H_n(X,A)$

If $\bar{\alpha} \in C_n(X,A)$ with $\partial(\bar{\alpha}) = 0 \in C_{n-1}(X,A)$ we may represent $\bar{\alpha}$ with $\alpha \in C_n(X)$ such that $\overline{\partial \alpha} = 0 \in C_{n-1}(X,A)$ that is $\alpha \in C_n(X)$ and $\partial\alpha \in C_n(A)$. If $\bar{\alpha} = \partial\bar{\beta}$ with $\bar{\beta} \in C_{n+1}(X,A)$, we may represent $\bar{\alpha}$ with $\alpha \in C_n(X)$, $\bar{\beta}$ with $\beta \in C_{n+1}(X)$ such that $\alpha = \partial\beta + \gamma$ for some $\gamma \in C_n(A)$.

Now we want to show that there is an exact sequence

$$(*)\quad \cdots \to H_n(A) \to H_n(X) \to H_n(X,A) \to H_{n-1}(A) \to H_{n-1}(X) \to \cdots$$

Here $H_n(A) \to H_n(X)$ is induced by inclusion $i: A \to X$ and $H_n(X) \to H_n(X,A)$ is induced by the quotient map $j: C_n(X) \to C_n(X)/C_n(A) = C_n(X,A)$ which gives a chain map and thus a map in homology.

The long exact sequence $(*)$ is an example of a general construction in homological algebra: Assume that we have three chain complexes $\mathcal{A} = (A_n)$, $\mathcal{B} = (B_n)$, $\mathcal{C} = (C_n)$ and for each n a short exact sequence

$$0 \to A_n \xrightarrow{i} B_n \xrightarrow{j} C_n \to 0$$

such that all squares in the diagram below commutes

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A_{n+1} \arrow[r, "\partial"] \arrow[d, "i"] & A_n \arrow[r, "\partial"] \arrow[d, "i"] & A_{n-1} \arrow[r] \arrow[d, "i"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "\partial"] \arrow[d, "j"] & B_n \arrow[r, "\partial"] \arrow[d, "j"] & B_{n-1} \arrow[r] \arrow[d, "j"] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial"] \arrow[d] & C_n \arrow[r, "\partial"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

So i and j are chain maps.
We say that the diagram is a short exact sequence of chain complexes.
Now it is a theorem that given such a short exact sequence of

(**) $$\to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \to \cdots$$

Here i_*, j_* are induced by the chain maps i, j and $\partial : H_n(C) \to H_{n-1}(A)$ is defined as follows:
Let $\bar{c} \in H_n(C)$, $c \in C_n$ with $\partial c = 0$.
Since j is surjective, $\exists$ $b \in B_n$ such that $j(b) = c$. Now, $j\partial b = \partial j b = \partial c = 0$ so $\partial b \in \ker j$,
and ~~so~~ by exactness $\partial b = ia$ with $a \in A_{n-1}$.
Also $i\partial a = \partial i a = \partial\partial b = 0$ and i is injective
so $\partial a = 0$, So a defines a homology class
$\bar{a} \in H_{n-1}(A)$, and we put $\partial \bar{c} = \bar{a}$.
One must check that ∂ is welldefined (independent of all choice of representatives)
and finally that (**) is exact.
(that is theorem 2.16 in Hatcher)
To do this you chase around in the diagram on page 6. (Try to do it yourself, it is not difficult.)

Now, considering the short exact sequence of chain complexes

$$0 \to C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X,A) \to 0$$

we get induced a long exact sequence in homology

$$\to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to$$

If we have a triple (X, A, B) of a space and subspace such that $B \subset A \subset X$ we have another short exact sequence of chain complexes induced by inclusions

$$0 \to C_n(A,B) \to C_n(X,B) \to C_n(X,A) \to 0$$

and again this induces a long exact sequence in homology

$$\to H_n(A,B) \to H_n(X,B) \to H_n(X,A) \to H_{n-1}(A,B) \to$$

Another example is to consider the short exact sequence

$$0 \to C_n(A) \to C_n(X) \to C_n(X,A) \to 0$$

for each $n \geq 0$ and adding in dimension -1 the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{1\!\!1} \mathbb{Z} \to 0 \to 0$$

So recalling $\varepsilon : C_0(A) \to \mathbb{Z}$ and $\varepsilon : C_0(X) \to \mathbb{Z}$ given by $\varepsilon(\sum n_i \sigma_i) = \sum n_i$ we get a diagram with a bottom lines

$$\begin{array}{ccccccccc}
0 & \to & C_0(A) & \to & C_0(X) & \to & C_0(X,A) & \to & 0 \\
 & & \downarrow \varepsilon & & \downarrow \varepsilon & & \downarrow & & \\
0 & \to & \mathbb{Z} & \xrightarrow{1\!\!1} & \mathbb{Z} & \longrightarrow & 0 & \to & 0
\end{array}$$

So this will induce a long exact sequence in reduced homology

$$\to \tilde{H}_n(A) \to \tilde{H}_n(X) \to H_n(X,A) \to \tilde{H}_{n-1}(A) \to$$

(Here A is assumed to be different from $\emptyset$)

Example

Consider $X = D^n$, $A = \partial D^n = S^{n-1}$

Then $\tilde{H}_i(D^n) = 0$ (since D^n is contractible)

so we get from the sequence in reduced homology:

$\to \cancel{0 \to \tilde{H}_{i+1}(D^n) \xrightarrow{\partial} \tilde{H}_i(S^{n-1})}$

$$0 = \tilde{H}_i(D^n) \to H_i(D^n, S^{n-1}) \xrightarrow{\partial} \tilde{H}_{i-1}(S^{n-1}) \to \tilde{H}_{i-1}(D^n) = 0$$

$$\text{So } H_i(D^n, S^{n-1}) \approx \tilde{H}_{i-1}(S^{n-1}) = \begin{cases} \mathbb{Z} & \text{for } i = n \\ 0 & \text{otherwise} \end{cases}$$

Example Consider the pair (X, x_0), $x_0 \in X$

Then

$$0 = \tilde{H}_i(x_0) \to \tilde{H}_i(X) \to H_i(X, x_0) \to \tilde{H}_{i-1}(x_0) = 0$$

So $\tilde{H}_i(X) \approx H_i(X, x_0)$ for all i.

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Note that ~~the prism~~ given maps $f, g : (X, A) \to (Y, B)$ between pairs, if there is a homotopy h_t with $h_t(A) \subset B$ $\forall t$ and $h_0 = f$, $h_1 = g$ then $f_* = g_* : H_n(X, A) \to H_n(Y, B)$ (see prop 2.19).

Excision

The excision theorem tell us that we can excise (cut away) certain subset of a space without changing the homology. So it will be a tool for calculating homology. The theorem is the following:

(Theorem 2.20) Given $Z \subset A \subset X$ such closure$(Z) \subset$ interior(A). Then the inclusion $(X-Z, A-Z) \to (X, A)$ induces an isomorphism, $H_n(X-Z, A-Z) \approx H_n(X, A)$ for each n (Z is excised from X and A)

An equivalent formulation is:
Given $A, B \subset X$ such that $\text{int}(A) \cup \text{int}(B) = X$ then the inclusion $(B, A \cap B) \to (X, A)$ induces an isomorphism $H_n(B, A \cap B) \approx H_n(X, A)$ for each n (see Hatcher for explanation of the equivalence of the two formulations)

Let X be a space $\mathcal{U} = \{U_j : j \in J\}$ be a collection of subspaces such that

$$X = \bigcup_{j \in J} \mathrm{int}(U_j).$$ Let $C_n^{\mathcal{U}}(X) \subset C_n(X)$

be the subgroup defined by $c = \sum n_i \sigma_i \in C_n^{\mathcal{U}}$ if each $\sigma_i \in C_n(U_j)$ for some $j \in J$.

Clearly if $c \in C_n^{\mathcal{U}}(X)$ then $\partial c \in C_{n-1}^{\mathcal{U}}(X)$ so $\{C_n^{\mathcal{U}}\}$ is a complex and we can define homology groups $H_n^{\mathcal{U}}(X)$.

Then the following proposition is essential in the proof of Theorem 2.20

<u>Prop 2.21</u> Let $i : C_n^{\mathcal{U}}(X) \to C_n(X)$ be the inclusion. Then $\exists$ $\rho : C_n(X) \to C_n^{\mathcal{U}}(X)$ a chain map such that $i\rho$ and ρi are chain homotopic to the identity. It follows that $i_* : H_n^{\mathcal{U}}(X) \to H_n(X)$ is an isomorphism.

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The proof of this, and the proof of 2.20 occupy almost 5 pages in Hatcher. I will write a short note explaining and commenting ~~the pr th~~ the proof later.