Solution of problem 0.24

Let us recall some definitions

Join: The join $X*Y$ of two spaces is the quotient space $X\times Y\times I/\sim$ where $(x,y_1,0)\sim(x,y_2,0)$ for $x\in X$, $y_1,y_2\in Y$ and $(x_1,y,1)\sim(x_2,y,1)$ for $x_1,x_2\in X$ and $y\in Y$.

Smashproduct $X\wedge Y$ of two spaces: Consider $X\times\{y_0\}, \{x_0\}\times Y\subset X\times Y$ where $x_0\in X$ and $y_0\in Y$. Forming the wedge $X\times\{y_0\}\vee\{x_0\}\times Y$ identifying (x_0,y_0) (which is a point in both spaces), we can think of $X\times\{y_0\}\vee\{x_0\}\times Y = X\vee Y$ as a subspace of $X\times Y$ and we define $X\wedge Y$ as the quotient space $X\times Y/X\vee Y$.

Suspension: $SX = X\times I/\sim$ where $(x_1,0)\sim(x_2,0)$ for $x_1,x_2\in X$ and $(x_1,1)\sim(x_2,1)$ for $x_1,x_2\in X$.

We want to show that $X*Y/X*\{y_0\}\cup\{x_0\}*Y$ and $S(X\wedge Y)/S(\{x_0\}\wedge\{y_0\})$ are homeomorphic. Here X,Y is CW-complexes and x_0,y_0 0-cells in X and Y respectively.

Consider first $X*Y/X*\{y_0\}\cup\{x_0\}*Y$.

This is the quotient space of $X\times Y\times I$ with respect to the equivalence relation which is generated by the following relations:

I a $(x, y_1, 0) \sim (x, y_2, 0) \quad \forall\, x, y_1, y_2$

I b $(x_1, y, 1) \sim (x_2, y, 1) \quad \forall\, x_1, x_2, y$

I c $(x_1, y_0, t_1) \sim (x_2, y_0, t_2) \sim (x_0, y_1, s_1) \sim (x_0, y_2, s_2) \quad \forall\, x_1, x_2, y_1, y_2, s_i, t_i \quad i = 1,2$

Here Ia, Ib are the relations that ~~defines~~ defining $X * Y$ and Ic are relations that identify all points in $X \times \{y_0\} \times I \cup \{x_0\} \times Y \times I$ and consequently also identity all points in $X \times \{y_0\} \cup \{x_0\} \times Y$ inside $X * Y$.

Now consider $S(X \wedge Y) / S(\{x_0\} \wedge \{y_0\})$

Again this space can be obtained as a quotient space of $X \times Y \times I$ with respect to the equivalence relation generated by the following relations.

II a $(x_0, y_1, t) \sim (x_0, y_2, t) \quad \forall\, y_1, y_2, t$

II b $(x_1, y_0, t) \sim (x_2, y_0, t) \quad \forall\, x_1, x_2, t$

(Note that in $X \wedge Y$, which is a quotient space of $X \times Y$ the points in $X \vee Y = X \times \{y_0\} \vee \{x_0\} \times Y$ are identified so in $S(X \wedge Y)$ which is a quotient space of $X \wedge Y \times I$ we must have the identifications given by IIa and IIb.)

II c $(x_1, y_1, 0) \sim (x_2, y_2, 0)$

II d $(x_1, y_1, 1) \sim (x_2, y_2, 1) \quad \forall\, x_1, x_2, y_1, y_2$

(This is identifications we get forming $S(X \wedge Y)$ as a quotient space of $X \wedge Y \times I$)

$\mathrm{II}e \quad (x_0, y_0, s) \sim (x_0, y_0, t) \quad \forall\, s, t$

(Since $S(\{x_0\} \wedge \{y_0\}) = (x_0, y_0) \times I$ we get $\mathrm{II}e$
by identifying all points in $S(\{x_0\} \wedge \{y_0\})$
inside $S(X \wedge Y)$).

To prove that the two quotient spaces are homeomorphic
it is enough to prove that $Ia), Ib), Ic)$
and $\mathrm{II}a, \mathrm{II}b, \mathrm{II}c, \mathrm{II}d, \mathrm{II}e$ generates the same
equivalence relation on $X \times Y \times I$.

We see that Ia and Ib are implied by $\mathrm{II}c$ and $\mathrm{II}d$.
Now consider Ic.
We have
$$(x_1, y_0, t_1) \overset{\mathrm{II}b}{\sim} (x_0, y_0, t_1) \overset{\mathrm{II}e}{\sim} (x_0, y_0, s_1) \overset{\mathrm{II}a}{\sim} (x_0, y_1, s_1)$$
Moreover
$$(x_0, y_0, t_1) \overset{\mathrm{II}e}{\sim} (x_0, y_0, t_2) \overset{\mathrm{II}b}{\sim} (x_2, y_0, t_2)$$
and
$$(x_0, y_0, t_1) \overset{\mathrm{II}e}{\sim} (x_0, y_0, s_2) \overset{\mathrm{II}a}{\sim} (x_0, y_2, s_2)$$
From this we see that the relations $\mathrm{II}a \cdots \mathrm{II}e$ implies Ic.
This implies that the equivalence relation generated
by $\mathrm{II}a \cdots \mathrm{II}e$ implies the —"————————— by
$Ia \cdots Ic$.
Next we will prove the opposite.
We see that the relations $\mathrm{II}a$ and $\mathrm{II}b$ are implied by Ic.

Now

$$(x_1,y_1,0) \overset{I_a}{\sim} (x_1,y_0,0) \overset{I_c}{\sim} (x_2,y_0,0) \overset{I_a}{\sim} (x_2,y_2,0)$$

so II_c is satisfied

Moreover

$$(x_1,y_1,1) \overset{I_b}{\sim} (x_0,y_1,1) \overset{I_c}{\sim} (x_0,y_2,1) \overset{I_b}{\sim} (x_2,y_2,1)$$

so II_d holds.

Finally II_e follows from I_c.

So the relation given by $II_a \cdots II_e$ is implied by the relation given by I_a, I_b, I_c.

So both spaces can be obtained as the same quotient space of $X \times Y \times I$.

From this we will now deduce that $X * Y \simeq S(X \wedge Y)$

Consider

$$X * \{y_0\} \cup \{x_0\} * Y = \frac{X \times \{y_0\} \times [0,1]}{X \times \{y_0\} \times \{1\}} \cup \frac{\{x_0\} \times Y \times I}{\{x_0\} \times Y \times \{0\}}$$

Since $X * \{y_0\}$ and $\{x_0\} * Y$ ~~are~~ thus are homeomorphic to the cones CX and CY respectively and also the intersection is homeomorphic to $\{x_0\} \times \{y_0\} \times [0,1]$, both spaces and their intersections are contractible.

Also since X, Y are CW-complexes with x_0, y_0 0-cells, it follows (from Hatcher) that all spaces constructed above also are CW-complexes (and subcomplexes)

By the conclusion of ex. 23 it follows that $X * \{y_0\} \cup \{x_0\} * Y$ is contractible.

So by prop. 0.17 the quotient map

$$X * Y \to X*Y / X*\{y_0\} \cup \{x_0\}*Y$$ is a homotopy-equivalence.

Now $S(\{x_0\} \wedge \{y_0\})$ is homeomorphic to I, so it is contractible

and therefore

$$S(X \wedge Y) \to S(X \wedge Y) / S(\{x_0\} \wedge \{y_0\})$$

is also a homotopy equivalence. Since the two quotient spaces we considered above were homeomorphic and therefore also homotopy-equivalent it follows (since homotopy-equivalence is an equivalence relation) that $X * Y$ and $S(X \wedge Y)$ are homotopy-equivalent spaces.