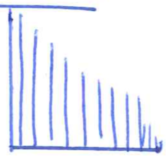
Exercises

6a)  $X = [0,1]\times\{0\} \cup \bigcup_{r\in Q\cap[0,1]} \{r\}\times[0,1-r]$

X deformation retracts to any point in $[0,1]\times\{0\}$, but not to any other point.

Proof

Define $f_t : X \to X$, $t\in[0,1]$ by

$$f_t(r,s) = (r,(1-2t)s) \text{ for } t\in[0,\tfrac{1}{2}],\ r\in Q\cap[0,1],\ s\in[0,1-r]$$

and $f_t(x,0) = (x,0)$, $x\in[0,1]$, $t\in[0,\frac{1}{2}]$.

Let $x_0\in[0,1]$, let $g_t(x,0) = ((-2t+2)x+(2t-1)x_0, 0)$

Let $h_t = \begin{cases} f_t & t\in[0,\frac{1}{2}] \\ g_t\circ f_{1/2}, & t\in[\frac{1}{2},1] \end{cases}$

Since $g_{1/2}(x,0) = (x,0)$, h_t is welldefined

and $h_0 = 1_X$, $h_1(y) = (x_0,0)$ for all $y\in X$

Also $h_t(x_0,0) = (x_0,0)$ for all t.

Hence X deformation retracts to $(x_0,0)$.

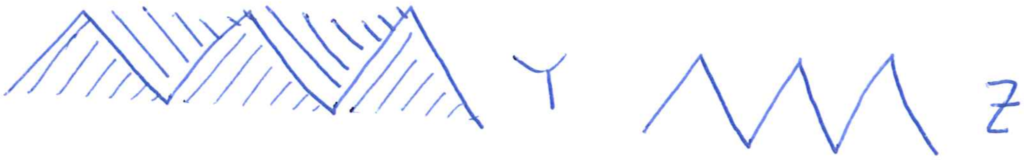
Let us assume that $y_0 = (r_0, s_0)$ with $r_0 \in \mathbb{Q} \cap [0,1]$ and $s_0 \in (0, 1-r_0)$. To get a contradiction assume that we have maps $f_t : X \to X$ s.t. $f_0 = \mathbb{1}_X$, $f_0(y) = y_0$ $\forall y \in X$ and $f_t(y_0) = y_0$ $\forall t \in [0,1]$.

We may find a small disc U around y_0 s.t. $\hat{U} = U \cap X$ is not path-connected.

Let $y = (r,s) \in \tilde{U}$ with $r \neq r_0$. We claim that if y is chosen sufficiently close to y_0 then $f_t(y) \in \hat{U}$ for all $t \in [0,1]$. If not we may find $y_n = (r_n, s_n) \in \tilde{U}$, with $r_n \to r_0$, $r_n \neq r_0$ and t_n such that $f_{t_n}(y_n) \notin \tilde{U}$. Since $[0,1]$ is compact, we may, by redefining our sequence, assume that $t_n \to t_0$ for some $t_0 \in [0,1]$.

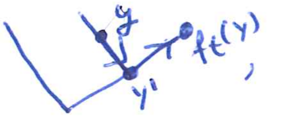
Now by continuity of $f_t(y)$ in y and t we get that $f_{t_n}(y_n) \to f_{t_0}(y_0) = y_0$, but this is impossible since $f_{t_n}(y_n) \in \hat{U}$ and $\hat{U}$ is a neighbourhood of y_0. So if y is close to y_0 $f_t(y) \in \tilde{U}$ for all t and $t \to f_t(y)$ is a path in $\hat{U}$ from y to y_0. Since $y = (r,s)$ with $r \neq r_0$ y and y_0 belongs to different path components in $\hat{U}$ so no such path exists in $\hat{U}$.

This gives a contradiction.

b) 

Let $y \in Y$, we will define a homotopy

$f_t : Y \to Y$ for $t \in [0, \frac{1}{2}]$.

For $y \in Y - Z$ 

, define $f_t(y)$ by sliding y along the line y belongs to in $Y - Z$ with a speed $2t$ until you possibly reach a point y' in Z and then slide along Z to the ~~white~~ right with same speed $2t$, you stop when you have traveled a ~~time~~ unit of time (so if t is small you may stay in $Y - Z$ for the whole time interval $[0, 1]$)

For $y \in Z$ slide to the right a unit of time with speed $2t$.

Now $f_0 = 1\!\!1_Y$ and $f_{1/2}(Y) = Z$. Furthermore

$f_t(Z) = Z$, but $f_t(z) \neq z$ for all $z \in Z$ and $t > 0$.

Let $z_0 \in Z$, since Z is contractible $\exists$

$g_t : Z \to Z$, $t \in [\frac{1}{2}, 1]$ with $g_{1/2} = 1\!\!1_Z$, $g_1(z) = z_0$

$\forall z \in Z$, let $h_t : Y \to Y$ be defined by

$$h_t = \begin{cases} f_t & t \in [0, \frac{1}{2}] \\ g_t \circ f_{1/2}, & t \in [\frac{1}{2}, 1] \end{cases}$$

Then h_t is a contraction of Y to the point z_0.

To see that Y does not deformation retract to any point, we observe that any point of Y has a neighbourhood which is not path-connected. This allow us to argue exactly as in 6a)

c) The map h_t we constructed in b) is a deformation retraction to Z in the weak sense (since $h_1(X) = \{z_0\} \subset Z$ and $h_t(Z) = Z$ for all t, (but $h_t|Z \neq 1\!\!1_Z$))