Lecture 30/3-2020

$\Delta^n = \{(x_0, \dots, x_n) \in \mathbb{R}^n \mid x_i \geq 0, \sum_{i=0}^{n} x_i = 1\}$

$\Delta^n \subset \mathbb{R}^{n+1}$ standard simplex

Let X be a topological space

$\sigma : \Delta^n \to X$ a (continuous) map

σ is a singular n-simplex in X

$C_n(X)$ free abelian group generated by all singular simplicies

$c \in C_n(X)$, $c = \sum_i n_i \sigma_i$, $n_i \in \mathbb{Z}$, σ_i singular simplex, a formal finite sum.

Let $v_0 = (1, 0, \dots, 0), \dots, v_n = (0, \dots, 1)$

$v_0, \dots, v_n \in \mathbb{R}^{n+1}$, $\{v_0, \dots, v_n\}$ standard basis in $\mathbb{R}^{n+1}$

Let $\{v_0', \dots, v_{n-1}'\}$ be the standard basis i $\mathbb{R}^n$

$[v_0, \dots, v_{i-1}, \hat{v}_i, \dots, v_n]$, denotes the $n-1$ simplex in Δ^n given by the linear map $\Delta^{n-1} \to \Delta^n$ such that

$v_0' \to v_0, v_1' \to v_1, \dots, v_{i-1}' \to v_{i-1}, v_i' \to v_{i+1}, \dots, v_{n-1}' \to v_n$

Now if we compose this map with e map $\sigma : \Delta^n \to X$ we get a map $\Delta^{n-1} \to X$ an $n-1$ simplex

We denote this by $\sigma|[v_0, \dots, \hat{v}_i, \dots, v_n]$

Let $\sigma : \Delta^n \to X$, define

$$\partial_n \sigma = \sum_{i=0}^{n} (-1)^i \sigma | [v_0, \ldots, \hat{v}_i, \ldots, v_n] \in C_n(X)$$

If $c = \sum_j n_j \sigma_j \in C_n(X)$, we define $\partial_n c = \sum_j n_j \partial_n \sigma_j$

So $\partial_n : C_n(X) \to C_{n-1}(X)$

Similar to simplicial homology $\partial_{n-1}\partial_n = 0$

We get a complex

$$\to C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \to$$

and define singular homology

$$H_n(X) = \ker \partial_n / \operatorname{Im} \partial_{n+1}$$

$(\operatorname{Im} \partial_{n+1} \subset \ker \partial_n$ since $\partial_n \partial_{n+1} = 0)$

Prop 2.6

$X = \bigcup_\alpha X_\alpha$ a space, $\{X_\alpha\}$ path-components of X. Then $H_n(X) \approx \bigoplus_\alpha H_n(X_\alpha)$

Proof

We have $C_n(X) = \bigoplus_\alpha C_n(X_\alpha)$

and $\partial_n(C_n(X_\alpha)) \subset C_{n-1}(X_\alpha)$

and the ~~statement~~ conclusion follows from this.

Prop 2.7

$X \neq \phi$, and path connected. Then $H_0(X) \approx \mathbb{Z}$.

If $X = \bigcup_\alpha X_\alpha$, each X_α - path component then $H_0(X) \approx \bigoplus_\alpha \mathbb{Z}_\alpha$, $\mathbb{Z}_\alpha$ copy of $\mathbb{Z}$

Sketch of proof: $\partial_0 : C_0(X) \to \{0\}$, so $\partial_0 = 0$

$H_0(X) = C_0(X)/_{\mathrm{Im}\,\partial_1}$. A zero simplex is a map $\{1\} \to X$. Let $c = \sum_i n_i b_i \in C_0(X)$. Define $\varepsilon : C_0(X) \to \mathbb{Z}$, $\varepsilon(c) = \sum_i n_i$. One can prove that $\mathrm{Im}\,\partial_1 = \ker \varepsilon$, ~~so~~ and ε is surjective. So

$\mathbb{Z} \approx C_0(X)/_{\ker \varepsilon} = C_0(X)/_{\mathrm{Im}\,\partial} = H_0(X)$.

What is the homology of a point $X=\{x\}$?

There is only one ~~si~~ n simplex $\sigma:\Delta^n \to X$, $\sigma(p)=x\ \forall p\in\Delta^n$.

$C_n(X)\approx\mathbb{Z}$ for all n.

What is $\partial_n: C_n(X)\to C_{n-1}(X)$

$\sigma:\Delta^n\to X$, $\partial_n(\sigma)=\sum_{i=0}^{n}(-1)^i\sigma|[v_0,..,\hat{v_i},..,v_n]$ ⊛

Since there is only one $n-1$ simplex

$\sigma|[v_0,..,\hat{v_i},..,v_n]=\sigma|[v_0,,\hat{v_j},..,v_n]\ \forall i,j$

⊛ has $n+1$ terms. So $\partial_n(\sigma)=0$ when $n+1$ is even and n odd, and $\partial_n(\sigma)=$ the $n-1$-simplex in $C_{n-1}(X)$

when n is even.

So when $n\overset{>0}{}$ is even we have,

$$C_{n+1}(X)\xrightarrow[\partial_{n+1}]{0}C_n(X)\xrightarrow[\approx]{\partial_n}C_{n-1}$$

So $\ker\partial_n=0$, and $\ker\partial_n/\operatorname{Im}\partial_{n+1}=\{0\}$

When $n>0$ is odd, we hav

$$C_{n+1}\xrightarrow[\approx]{\partial_{n+1}}C_n(X)\xrightarrow[\partial_n]{0}C_{n-1}$$

So $\operatorname{im}\partial_{n+1}=C_n(X)=\ker\partial_n$ and $H_n(X)=\dfrac{C_n(X)}{C_n(X)}=\{0\}$.

<u>Prop 2.8</u> If $X = \{x\}$, then $H_n(X) = 0$ for $n > 0$ and $H_0(X) \cong \mathbb{Z}$

It seems however somewhat unnatural that a one-point space does not have trivial homology in all dimension.

Recall the map $\varepsilon : C_0(X) \to \mathbb{Z}$. It is easy to see that $\varepsilon(\sum_i n_i b_i) = \sum_i n_i$. So we have a complex $\varepsilon \partial_1 = 0$.

$$\to C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

and we can take the homology of this complex. We denote the homology groups by $\tilde{H}_n(X)$ (reduced homology)

Clearly $\tilde{H}_n(X) = H_n(X)$ for $n > 0$.

What is $\tilde{H}_0(X)$?

We have $\varepsilon : C_0(X) \to \mathbb{Z}$. This map is surjective $\operatorname{im}\partial_1 \subset \ker \varepsilon$, and we get $\bar{\varepsilon} : \frac{C_0(X)}{\operatorname{im}\partial_1} = H_0(X) \to \mathbb{Z}$

$\ker \bar{\varepsilon} = \frac{\ker \varepsilon}{\operatorname{im}\partial_1} = \tilde{H}_0(X)$, so we get an

exact sequence

$$0 \to \tilde{H}_0(X) \to H_0(X) \xrightarrow{\bar{\varepsilon}} \mathbb{Z} \to 0$$

Since $\mathbb{Z}$ is free, the sequence splits and we get $H_0(X) \simeq \tilde{H}_0(X) \oplus \mathbb{Z}$

Now if X is path connected $H_0(X) \simeq \mathbb{Z}$ hence $\tilde{H}_0(X) = \{0\}$.

Especially $\tilde{H}_n\{x\} = 0$ for all n.

In general let

$$\begin{array}{ccccccccc}
\xrightarrow{\partial_{n+2}} & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \xrightarrow{\partial_{n-1}} \\
& \downarrow \alpha_{n+1} & & \downarrow \alpha_n & & \downarrow \alpha_{n-1} & \\
\xrightarrow{\partial} & D_{n+1} & \xrightarrow{\partial_{n+1}} & D_n & \xrightarrow{\partial_n} & D_{n-1} & \xrightarrow{\partial_{n-1}}
\end{array}$$

be two chain complexes of abelian groups

(where $\partial \circ \partial = 0$)

~~Assume~~ we have ~~maps~~ homomorphisms

$\alpha_n : C_n \to D_n$ for each n

Assume that the diagram commutes

that is
$$\begin{array}{ccc}
C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n \\
\alpha_{n+1} \downarrow & \partial & \downarrow \alpha_n \\
D_{n+1} & \xrightarrow{\partial_{n+1}'} & D_n
\end{array}$$

$\alpha_n \circ \partial_{n+1} = \partial_{n+1} \alpha_{n+1}$

~~$\alpha_n \circ \partial = \partial_{n+1}$~~ (for each n)

Note that for each of these complexes we can define homology-groups. $\frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}}$

Now if the diagram commutes we get induced a map $\alpha_* : H_n(C) \to H_n(D)$

To see this:

Take $c \in C_n$ s.t. $\partial_n c = 0$. Let $[c]$ be the homology class. Now $\partial_n(\alpha_n(c)) = \alpha_{n-1}(\partial c) = \alpha_{n-1}(0) = 0$

So $\alpha_n(c)$ represent a homology class, and we put $\alpha_*([c]) = [\alpha_n(c)]$.

Note that if $c' = c + \partial_{n+1} c''$, $c'' \in C_{n+1}$

then $\alpha_n(c') = \alpha_n(c) + \alpha_n(\partial_{n+1} c'')$

$$= \alpha_n(c) + \partial_{n+1}(\alpha_{n+1} c'')$$

So $[\alpha_n(c')] = [\alpha_n(c)]$. So α_* is well defined.

So chain maps induces maps in homology.

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Application of this:

Let $f: X \to Y$ a map.

$\sigma: \Delta^n \to X$ a simplex in X

$f \circ \sigma: \Delta^n \to Y$ — " — in Y

Can define $f_\#: C_n(X) \to C_n(Y)$

$f_\#(c) = f_\#\left(\sum_i n_i \sigma_i\right) = \sum_i n_i f \circ \sigma_i$

Now since $f \circ (\sigma | [v_0, \dots, \hat{v}_i, \dots, v_n]) =$

$= (f \circ \sigma) | [v_0, \dots, \hat{v}_i, \dots, v_n]$

we get that $f_\#(\partial_n) = \partial_n \circ f_\#$

$$\begin{tikzcd} C_n(X) \arrow[r, "\partial_n"] \arrow[d, "f_\#"] \arrow[dr, phantom, "\circlearrowleft"] & C_{n-1}(X) \arrow[d, "f_\#"] \\ C_n(Y) \arrow[r, "\partial_n"] & C_{n-1}(Y) \end{tikzcd}$$

So $f_\#$ is a chain map and we get induced

$f_*: H_n(X) \to H_n(Y)$ for each n.

It is easy to see that if

$X \xrightarrow{f} Y \xrightarrow{g} Z$ then $(g \circ f)_* = g_* \circ f_*$

Also $X \xrightarrow{\mathbb{1}} X$ $(\mathbb{1})_* = \mathbb{1}$ (the identity)

So if f is a homeomorphism and $g = f^{-1}$

then $\mathbb{1} = (\mathbb{1}_X)_* = (g \circ f)_* = g_* \circ f_*$ and $\mathbb{1} = (\mathbb{1}_Y)_* = f_* \circ g_*$

So f_* is an <u>iso</u>morphism with $f_*^{-1} = g_*$

We also have the following important theorem

<u>Theorem 2.10</u>

If $f, g : X \to Y$ are homotopic then

$f_* = g_* : H_n(X) \to H_n(Y)$ for all n

This has the consequence.

<u>Corollary</u>

If $f : X \to Y$ is a homotopy-equivalence then $f_* : H_n(X) \to H_n(Y)$ is an isomorphism.

Now consider the complexes and map

$$\begin{tikzcd}
\xrightarrow{\partial} C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "f_\# \downarrow g_\#"'] & C_n(X) \arrow[r, "\partial"] \arrow[dl, "P"] \arrow[d, "f_\# \downarrow g_\#"'] & C_{n-1}(X) \xrightarrow{\partial} \arrow[dl, "P"] \arrow[d, "f_\# \downarrow g_\#"'] \\
\xrightarrow{\partial} C_{n+1}(Y) \arrow[r, "\partial"'] & C_n(Y) \arrow[r, "\partial"'] & C_{n-1}(Y) \xrightarrow{\partial}
\end{tikzcd}$$

Now it follows from the construction of P that $\partial P = g_\# - f_\# - P\partial$ and this will imply that $g_\#, f_\#$ induces the same maps in homology.

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This is a general fact from homological algebra.

Assume that we have two chain complexes, C, D two chain maps α, β and a ~~ma~~ homomorphism $P: C_n \to D_{n+1}$ (for each n) such that $\partial P = \beta - \alpha - P\partial$

$$\begin{tikzcd}
\to C_{n+1} \arrow[r, "\partial"] \arrow[d, "\alpha \downarrow \beta"'] & C_n \arrow[r, "\partial"] \arrow[dl] \arrow[d, "\alpha \downarrow \beta"'] & C_{n-1} \arrow[dl] \arrow[d, "\alpha \downarrow \beta"'] \\
D_{n+1} \arrow[r] & D_n \arrow[r] & D_{n-1}
\end{tikzcd}$$

then α, β induce the same maps in homology.

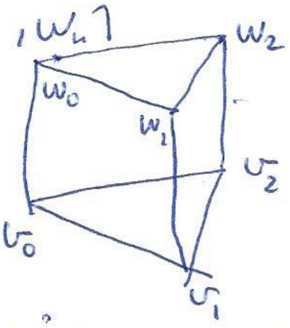
To prove it we chase around in the diagram, Some remarks about the proof of

Thm. 2.10

If $\sigma: \Delta^n \to X$ is a simplex, we get a map

$$\Delta^n \times I \xrightarrow{\sigma \times 1\!\!1} X \times I$$

Let the vertices at level 0 in $\Delta^n \times I$ be $[v_0, \dots, v_n]$ and the vertices at level 1 be $[w_0, \dots, w_n]$



$\Delta^2 \times I$

$$\sum_i (-1)^i [v_0, \dots, v_i, w_i, \dots, w_n]$$

is a $n+1$ chain in $\Delta^n \times I$

Now let $f \simeq g$ and $F: X \times I \to Y$ be a homotopy with $f_0 = f$, $f_1 = g$.

We have a map $F \circ \sigma \times 1\!\!1$

$$\Delta^n \times I \xrightarrow{\sigma \times 1\!\!1} X \times I \xrightarrow{F} Y$$

So $(F \circ (\sigma \times 1\!\!1))_\# \left(\sum_i (-1)^i [v_0, \dots, v_i, w_i, \dots, w_n] \right)$ is a $n+1$ chain in Y. Denoting this chain by $P(\sigma)$ we get a map $P: C_n(X) \to C_{n+1}(Y)$