MAT 4530 Algebraic Topology I

2020 Spring

Main reference :

Hatcher , "Algebraic Topology"

Lecture 1

- Introduction
- Homotopy (Chap. 0)

Intro : what is Algebraic Topology ?

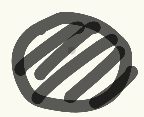
What to study : property of shapes

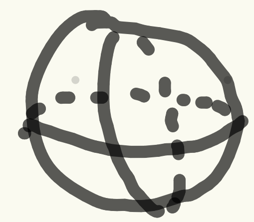
- allow continuous deformation

"everything is made of rubber"

⇝ do not care about distance,
homeomorphism type, but ..

... look at what cannot be changed (<u>invariant</u>) under continuos deform.

$\cdot \sim$  $\not\sim$ 

$\not\sim$ 

point disk ring sphere

ring: has a hole with (essentially) 1-dim boundary
or, has a loop

sphere: same with 2-dim bdry

How to study: count loops, holes, ..

using abstract algebra

↳ groups, rings, modules

→ make sense of "product" $gh \in G$ $g, h \in G$

"unit" $eg = g = ge$

"inverse" g^{-1}; $gg^{-1} = e$

→ make sense of $ab + c \in R$

for $a, b, c \in R$

→ make sense of $am + m' \in M$

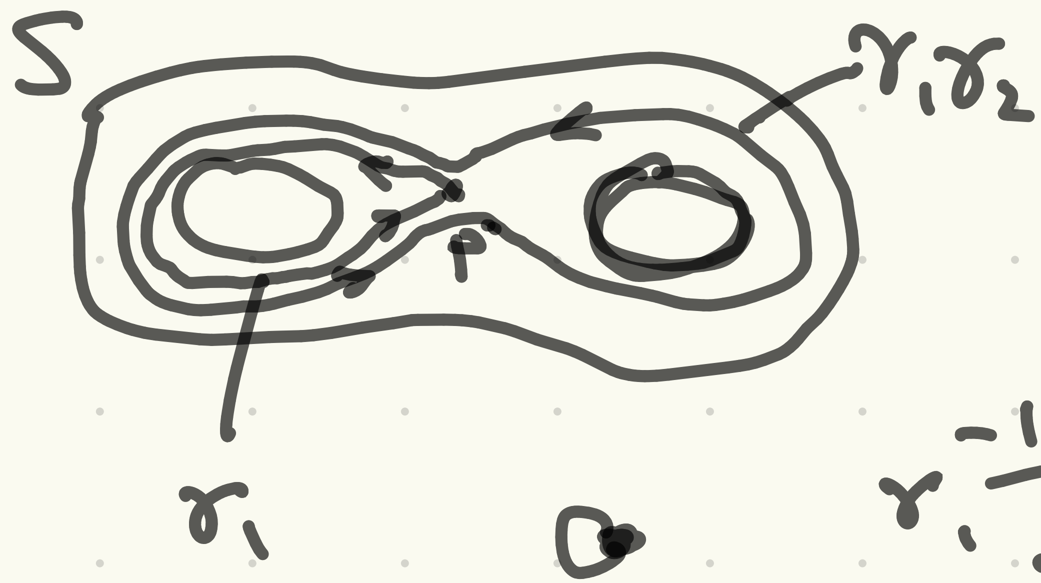
$a \in R$, $m, m' \in M$

⇝ $(ab+c)m = a(bm) + cm$

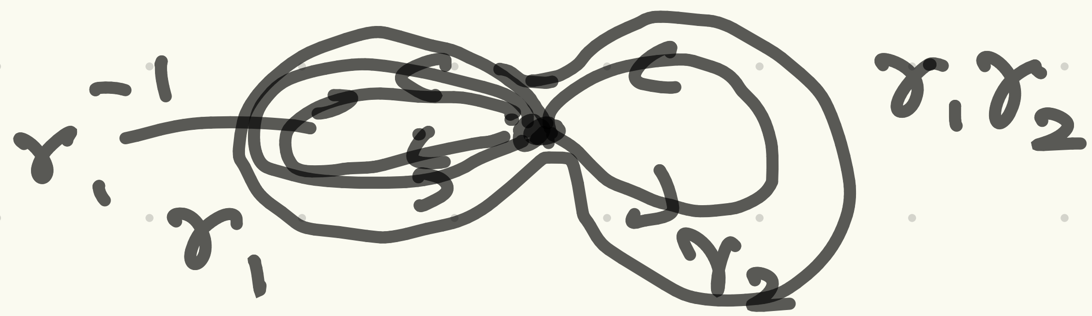
Main goal: two fundamental invariants

1. <u>fundamental</u> <u>group</u> $\pi_1(\underset{\text{space}}{S}, \underset{\text{base point}}{p})$

count loops in S starting (and ending) at p



group by ... <u>concatenation</u>



2. (singular) homology groups

$H_n(X)$ $\quad n = 0, 1, 2, \ldots$ (comm. groups)

count maps $S^n \xrightarrow{\sigma} X$ and linearize

i.e., make sense of $\sigma_1 + \sigma_2$, etc.

3. give computable models of π_1, H_n for spaces given by glueing basic parts

etc.

A look back: Cauchy's residue theorem

$f(z)$: meromorphic function

e.g. $f(z) = \frac{1}{z}$ on some domain $D \subset \mathbb{C}$

γ : loop (closed curve) in D
not passing poles of f

$$\Rightarrow \underbrace{\int_\gamma f(z)\, dz}_{\text{e.g. } \int \frac{dz}{z}} = 2\pi i \sum_{w} \underbrace{\mathrm{wind}(\gamma, w)}_{\in \mathbb{Z}} \underbrace{\mathrm{Res}(f, w)}_{\text{residue}}$$

w: pole of f inside γ

winding number

RHS: inv for cont. deform of γ in $D \setminus \{\text{poles of } f\}$

so $\gamma \mapsto \int_\gamma f(z)\,dz$ is a functional

on the group $\pi_1(D \setminus \{\text{poles}\}, p)$

hom $\pi_1(D \setminus \{\text{poles}\}, p) \to \mathbb{C}$

concat prod. additive group

Chapter 0

Homotopy, homotopy type

Basic idea: continuous deform.

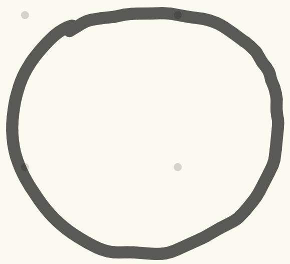
of shapes and maps

→ add extra continuous param. t

S_t , $f_t : X \to Y$, ...

Example $(0 \le t \le 1)$

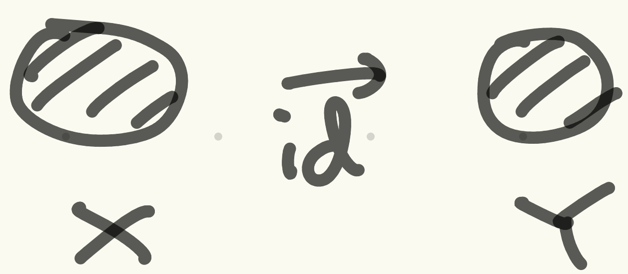
- $S_t = \{ (x,y) \in \mathbb{R}^2 ,$
$2-t \le \sqrt{x^2+y^2} \le 2+t \}$

S_0 : 

S_t ($0<t$) : 

$X = Y$

- $f_t(x,y) = \underline{(tx, ty)}$ for $\boxed{x^2+y^2 \le 1}$

f_1 : 

f_0 : 

Precise formulation

cont. deform. of maps

X, Y : topological spaces

$I = [0,1]$: unit interval

Def (homotopy, $f_0 \simeq f_1$) $\left(f_0 \underset{h}{\sim} f_1\right)$ etc.

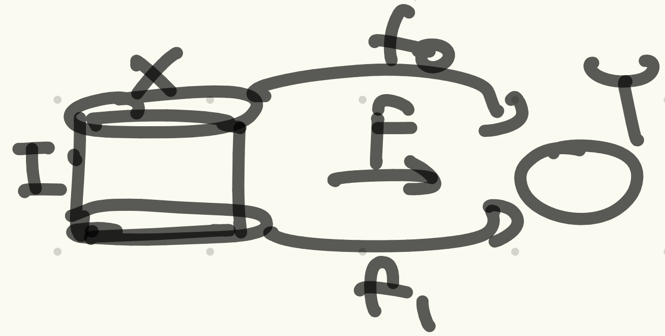
$f_0, f_1 : X \to Y$ cont. maps

are <u>homotopic</u> if $\exists\, F : X \times I \to Y$

cont. s.t. $f_i(x) = F(x, i)$ $x \in X$

$i = 0, 1$

(cont.) F : homotopy between f_0 and f_1



$\rightsquigarrow$ morally : continuous family of maps $f_t : X \to Y$ by

$$f_t(x) = F(x,t)$$

but we don't have to worry about the precise topology of

$$\{ f : X \to Y \ \text{cont} \} = Y^X$$

$(I \to Y^X)$

Def (retract ; shrinking down)

$A \subset X$ is a <u>deformation retract</u> if

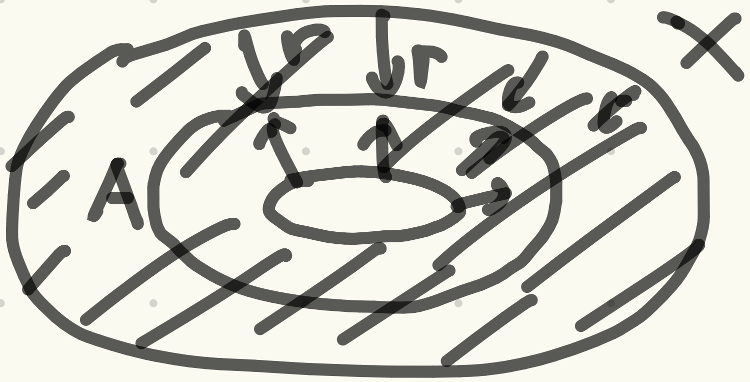
- $\exists\ r : X \to A$ (cont.) s.t.

  $r(a) = a$ for $a \in A$

- id_X and r are homotopic

  as cont. maps $X \to X$

($\exists\ F : X \times I \to X$, $F(x,0) = x$, $F(x,1) = r(x)$)



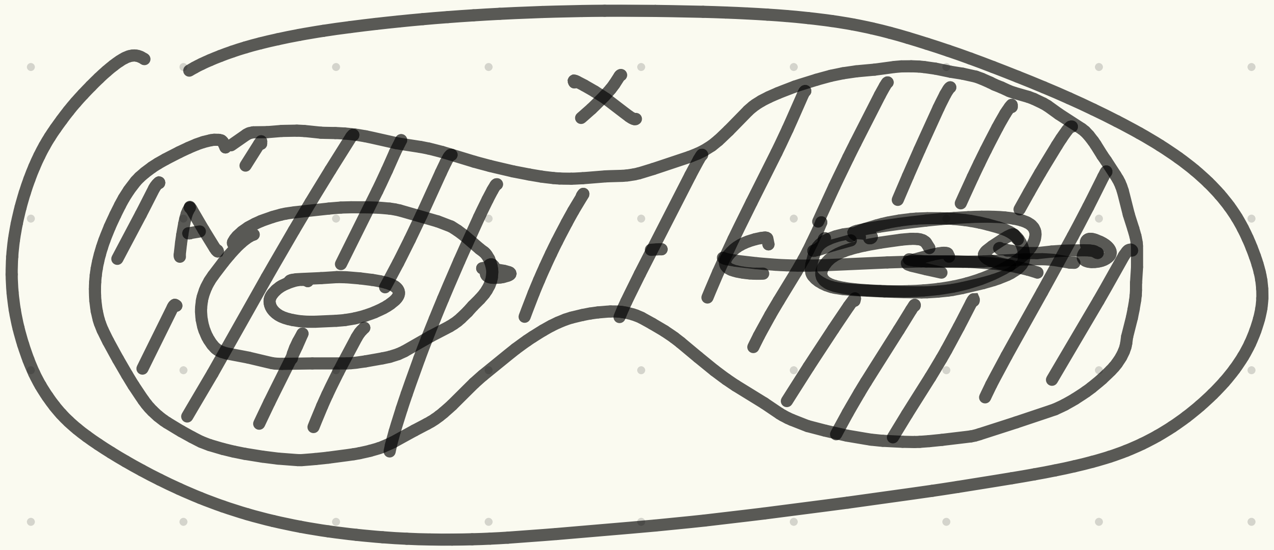
def. retv.

If $X = \{(x,y) : 1 \le \sqrt{x^2+y^2} \le 3\}$

$A = \{\sqrt{x^2+y^2} = 2\}$ $F((x,y),t) =$

$(1-t)r(x,y) + t(x,y)$

proj. from X to A



not def. retr.

($X \simeq_h Y$ etc.)

Def (homotopy equivalence $X \simeq Y$)

X and Y are <u>homotopy equivalent</u>

if $\exists$ $f : X \to Y$ s.t

$g : Y \to X$

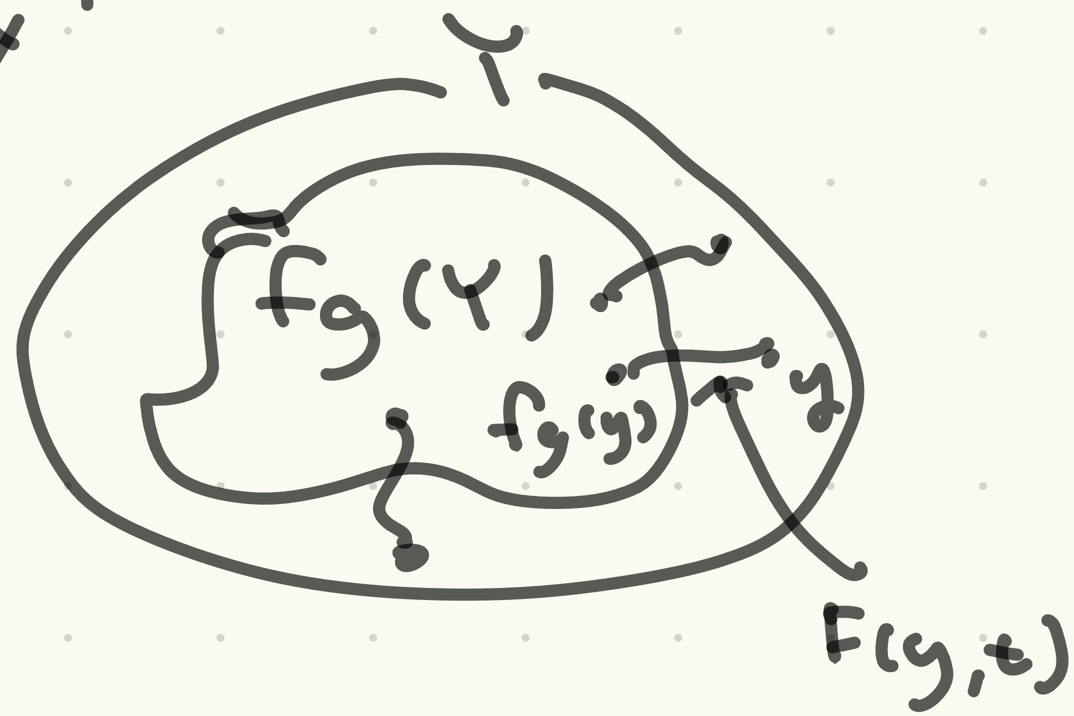
- fg and id_Y are homotopic
  as maps $Y \to Y$
- gf and id_X are homotopic
  as maps $X \to X$

$$X \underset{g}{\overset{f}{\rightleftarrows}} Y$$

f and g are inverses

"up to homotopy"

$fg \simeq id_Y$



Example : $A \subset X$ def. retract.

$\Rightarrow \quad A \simeq X$

- $i : A \to X$ inclusion map $i(a) = a$
- $r : X \to A$ retract map ($r(a) = a$ for $a \in A$)

$\rightsquigarrow \quad r\, i = id_A$, so nothing to do

$i\, r$ is homotopic to id_X by ($r : X \to X$) definition

Def. X is contractible if

$X \simeq pt$ top. sp. with one element

Ex. $D^k = \{(x_1, \ldots, x_k) : x_1^2 + \cdots + x_k^2 \leq 1\}$

$\subset \mathbb{R}^k$ (closed unit ball/disk)

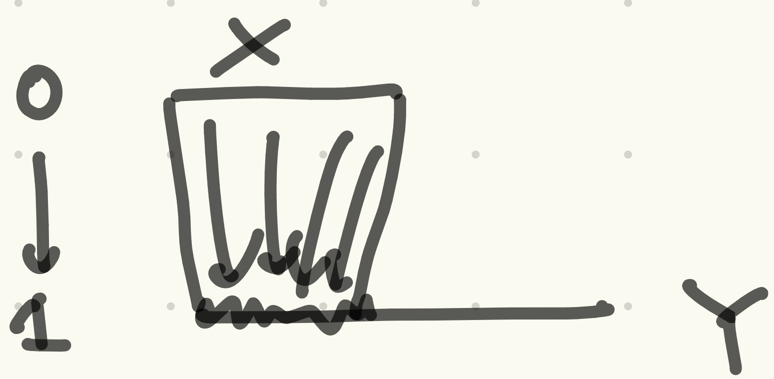
$A = \{(0, 0, \ldots, 0)\} \subset D^k$ is def. retr.

Advanced concept

<u>Mapping cylinder</u> for $f : X \to Y$

"glue" one end of $X \times I$ into Y using f

$$M_f : (X \times I) \amalg Y \,/\, (x, 1) \sim f(x)$$



(quotient sp. by this equiv. rel.)

Next : cell complexes

Basic idea: build spaces by glueing <u>basic</u> <u>parts</u>

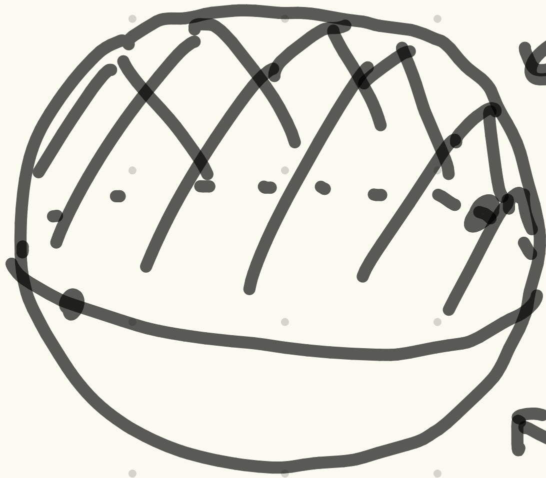
↳ copies of $D^k \subset \mathbb{R}^k$

Ex. S^1

$\cong$ ← homeomorphism

(cf. $\simeq$ homotopy equiv.

S^2 :



copy of D^2

S^1 = glueing of $\begin{cases} 2 \times D^1 \\ 2 \times D^0 \end{cases}$

another copy of D^2

glueing of $2 \times D^k$ $k = 0, 1, 2$

Eventually: compute invariants of X from how its basic parts (<u>cells</u>) are connected