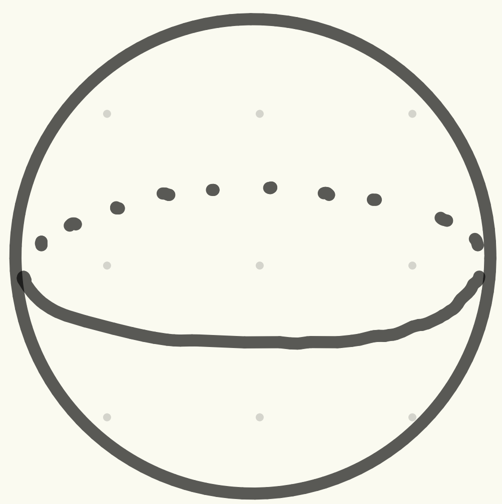
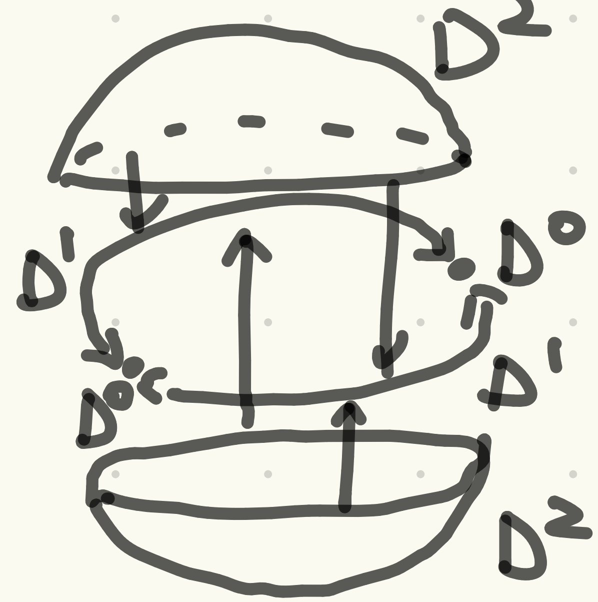
Cell complexes

Recall: we want to present shapes (top. sps.) by glueing copies of $D^k \subset \mathbb{R}^k$
closed unit disk

Ex. S^2



as



Def. A <u>cell complex</u> (<u>CW complex</u>) is a topological space X with following additional structure:

- subspaces $X^0 \subset X^1 \subset \cdots$ of X

  X^n: <u>n-skeleton</u> of X

- X^0 is discrete (by subsp. top.)
- X^n is given by "glueing" n-disks to X^{n-1}

  $\ldots$)

...) more precisely $\left(\alpha \in \underbrace{I_n}_{\text{ind. set}}\right)$

$\exists$ family of cont. maps $\varphi_\alpha : S^{n-1} \to X^{n-1}$

s.t. X^n is homeo. to the quot.

$\left(X^{n-1} \sqcup \left(\coprod_{\alpha \in I_n} D^n \right)\right) /$ if $x \in S^{n-1} \subset D^n$ at α-th comp.

$\uparrow$ disj union

then $x \sim \varphi_\alpha(x) \in X^{n-1}$

so $X^n \cong X^{n-1} \sqcup \left(\coprod_\alpha \mathring{D}^n \right)$ as sets

open disk

$\uparrow$

α-th n-cell of X

Recap:

X has subspaces $X^0 \subset X^1 \subset \cdots$

X^0 discrete

$\underset{n\text{-skeleton}}{X^n} = X^{n-1} \amalg \left(\coprod_{\alpha \in I_n} e^n_\alpha \right)$ as sets

$e^n_\alpha \cong \mathring{D}^n$ open n-disk

topology from glueing data

$\varphi_\alpha : S^{n-1} \to X^{n-1}$

- $(X^n)_{n=0}^{\infty}$ approximates X:

$X = \bigcup_{n=0}^{\infty} X^n$

$\exists\, V_n \subset X$

$V \cap X^n$

II

"

- $V \subset X$ open $\iff \forall n\ V \cap X^n \subset X^n$ open $V_n \cap X^n$

$\Rightarrow$ automatic

X is <u>finite</u> <u>dimensional</u> (as cell cplx)

if $\exists\, n \quad X = X^n$ (so $X^m = X^n \quad m \geq n$)

Nomenclature: CW complex

<u>c</u>losure-finiteness $\left[\begin{array}{l}\text{closure of each} \\ \text{cell } e^n_\alpha \subset \text{fin. num of cells}\end{array}\right.$

X has the <u>w</u>eak top. of $(X^n)_n$

Alternatively

def. cell complex as collection

of glueing data $\varphi_\alpha : S^{n-1} \to X^{n-1}$

X^n inductively defined as

$$X^n = X^{n-1} \amalg \left(\coprod_{\alpha \in I_n} D^n \right) / \text{equiv. rel as before}$$

$\rightsquigarrow$ define X as $X = \bigcup_{n=0}^{\infty} X^n$ as set

topology: wk top for $(X^n)_{n=0}^{\infty}$

Rephrasing the Definition (Prop A.2)

$$X^n \cong \left(X^{n-1} \amalg \left(\coprod_{\alpha \in A} D^n \right) \right) /$$ if $x \in S^{n-1} \subset D^n$ at α-th comp.

then $x \sim \varphi_\alpha(x) \in X^{n-1}$

gives cont. maps $\Phi_\alpha : D^n \xrightarrow{\text{inclu. to } \alpha\text{-th comp}} X^n \to X$ s.t.

(<u>characteristic</u> <u>map</u>)

(for $\alpha \in I_n \leftarrow$ index set)

- $\Phi_\alpha|_{\mathring{D}^n} : \mathring{D}^n \to \underbrace{\Phi_\alpha(\mathring{D}^n)}_{\alpha\text{-th } n\text{-cell}}$ is homeo

- $X = \coprod_{n=0}^{\infty} \coprod_{\alpha \in I_n} \Phi_\alpha(\mathring{D}^n)$ as sets

- $\Phi_\alpha(S^{n-1})$ is contained in

$$\Phi_{\beta_1}(D^{m_1}) \cup \cdots \cup \Phi_{\beta_k}(D^{m_k})$$

for some $m_1, \dots, m_k < n$, $\beta_i \in A^{m_i}$

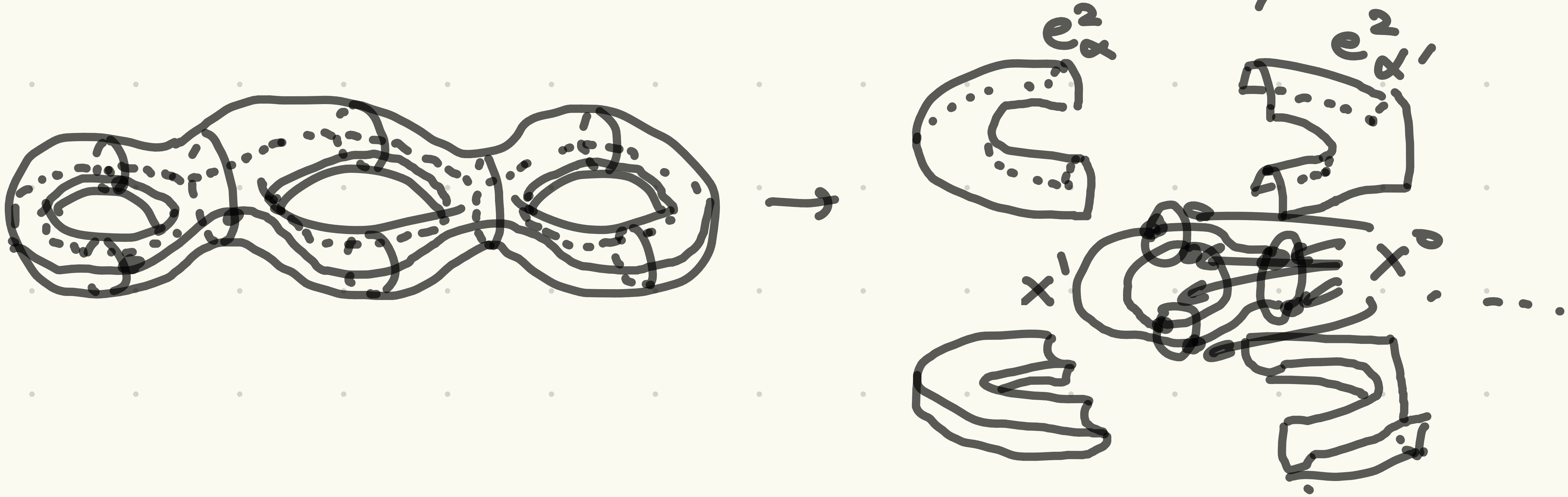
- $C \subset X$ is closed

$\Leftrightarrow$ $\forall n, \alpha \in A^n$: $\overline{C \cap \Phi_\alpha(\mathring{D}^n)} \subset X$ is closed

Some examples

- closed oriented surfaces

1. easier (but inefficient) way :

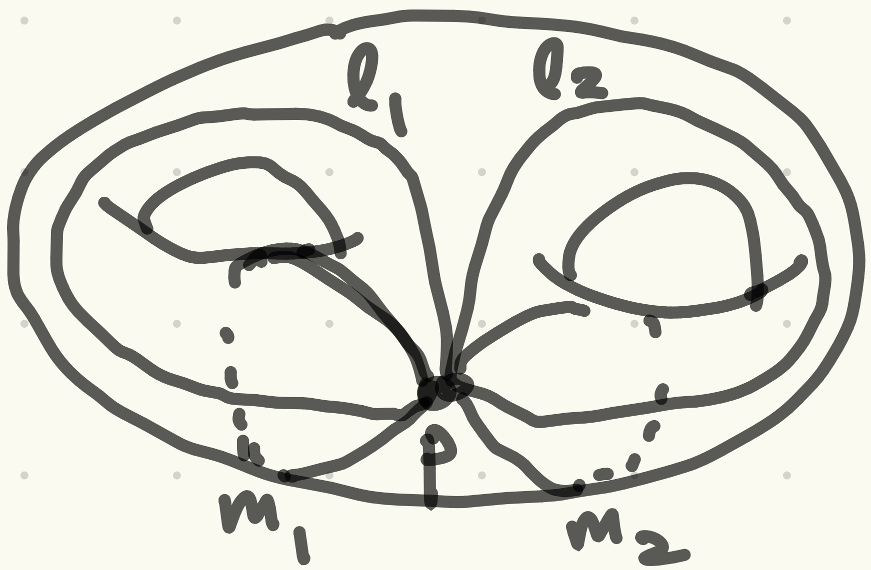


2. efficient (but telegraphic) way

fix a base point p $\quad X^0 = \{p\}$

take 2 loops ("meridian" and "longitude") through p for each hole $\rightsquigarrow$ 1-skeleton,

complement $\cong \mathring{D}^2$



- sphere S^n

one 0-cell and one n-cell

$X^0 = pt = X^1 = \cdots = X^{n-1} \subset X^n = S^n$

$\varphi : S^{n-1} \to pt$
$\quad\;\; \cap$
$\quad\; D^n$

cf. $S^n \cong D^n / S^{n-1}$ collapse $S^{n-1} \subset D^n$ to a point

What happens often:

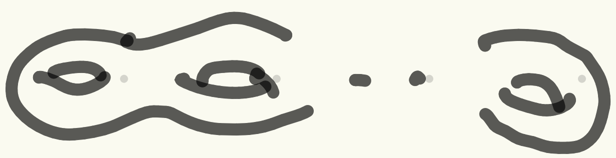
- want: compute invariants of X as a topological space
- easier to define / compute invariants using presentation as cell cplx
- need to check independence on the presentation

Example : Euler characteristic

X : 2-dim cell complex

$\rightsquigarrow \chi(X) = \#(2\text{-cells}) - \#(1\text{-cells}) + \#(0\text{-cells})$

$\chi(S^2) = 2, \quad \chi(S^1 \times S^1) = 0,$

$\chi\Big($  $\Big) = 2-2g$

$g \times$

Advanced : when do we care about ∞-dim cell complexes ?

The classifying space $B\underline{G}$ ← group

one 0-cell

$\underline{1}$-cells e^1_g $g \in G$

2-cells for each rel $gh = k$

$\cdots\cdots$

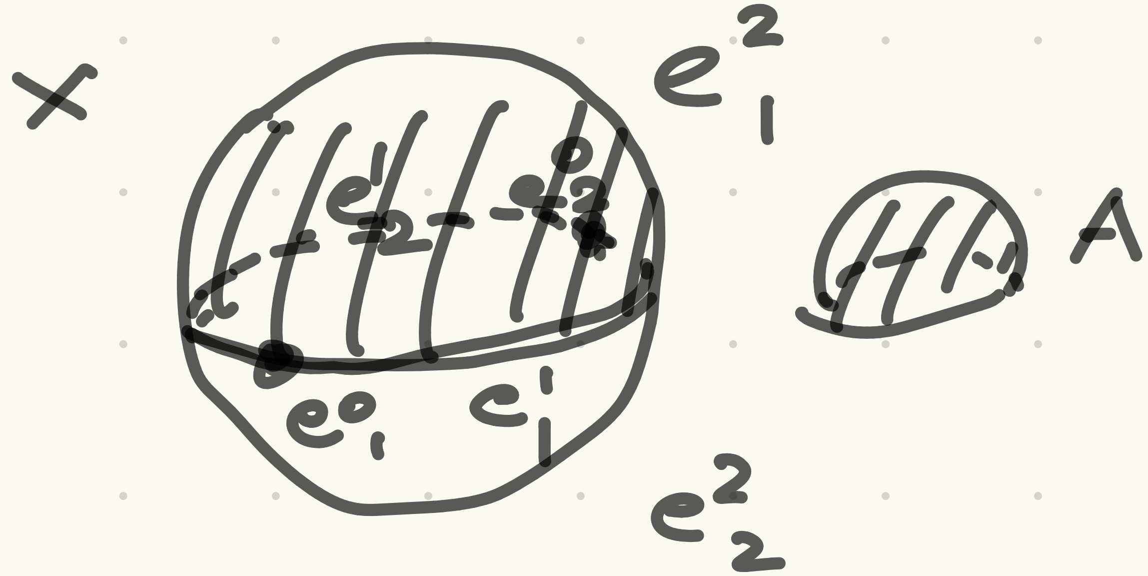
Def (subcomplex ; CW pair (X, A))

$X \left((X^n)_{n=0}^{\infty}, \varphi_\alpha : S^{n-1} \to X^{n-1}, \dots \right)$

cell complex ; $X = \coprod_{n=0}^{\infty} \coprod_{\alpha \in I_n} \Phi_\alpha(\mathring{D}^n)$

a <u>subcomplex</u> of X is a closed subset $A \subset X$ which is a union of cells of X

$A = \coprod_{n=0}^{\infty} \coprod_{\alpha \in J_n} \underbrace{\Phi_\alpha(\mathring{D}^n)}_{e_\alpha^n}$ with $J_n \subset I_n$



$e_1^2 \subset A$

$e_2^2 \cap A = \emptyset$

$e_1^1, e_2^1 \subset A$

$e_1^0, e_2^0 \subset A$

$e_1^2 \not\subset A$

but $e_1^2 \cap A \neq \emptyset$

! this depends on pres as cell cplx

$S^{n-1} \subset S^n$ subcplx if we take

$X^0 = S^0 \subset X^1 = S^1 \subset \cdots \subset X^n = S^n$

2 pts

but <u>not</u> if we take

$X^0 = pt = X^1 = \cdots = X^{n-1} \subset X^n = S^n$

Basic operations on cell complexes

Product of cell complexes

X, $\varphi_\alpha : S^{n-1} \to X^{n-1}$ $(\alpha \in I_n,\ n=1,2,\cdots)$

corresp. n-cell $e^n_\alpha \ (\cong \mathring{D}^n) \subset X^n$

Y, $\psi_\beta : S^{n-1} \to Y^{n-1}$ $(\beta \in J_n,\ n=1,2,\cdots)$

corresp. n-cell $e^n_\beta \ (\cong \mathring{D}^n) \subset Y^n$

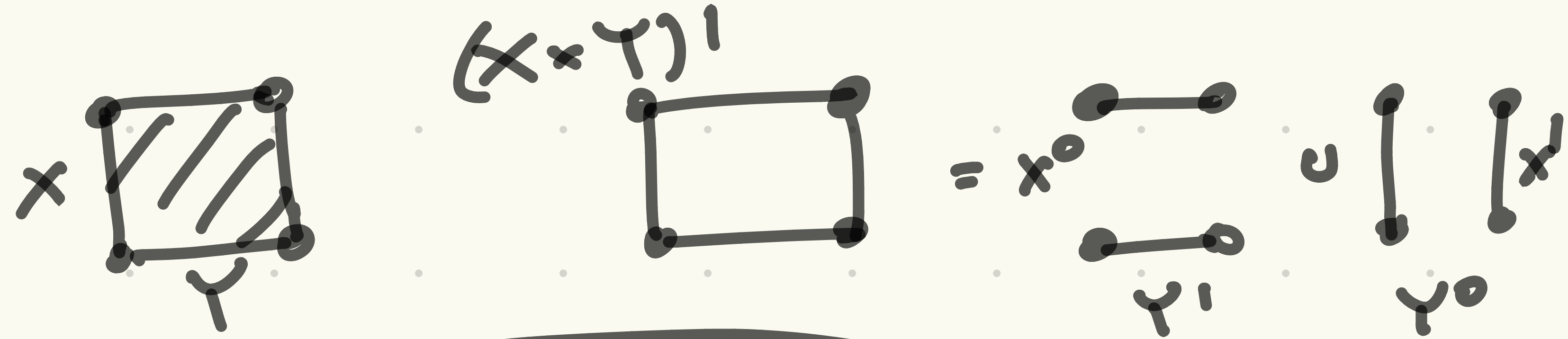
$\rightsquigarrow$ $X \times Y$ cell cplx with n-cells

$$e^m_\alpha \times e^{n-m}_\beta \ (\cong \mathring{D}^n) \subset X \times Y$$

$$\mathring{D}^m \cong (0,1)^m$$

$$\Rightarrow \mathring{D}^m \times \mathring{D}^{n-m} \cong (0,1)^n \cong \mathring{D}^n$$

Rem



- $(X\times Y)^n = \bigcup_{m=0}^{n} X^m \times Y^{n-m}$

- $U \subset X\times Y$ is open (for cell cplx topology)

$\iff$ $U \cap (X^m \times Y^{n-m})$ is open for $0 \le m \le n$

(finer than prod. top. of $X\times Y$)

Quotient by a subcomplex $A \subset X$

Basic idea: "collapse" A to a point

as set $(X \setminus A) \cup \{*\}$ often write $[A]$

as top. sp. X/A = quotient

of X by $a \sim a'$ for $a, a' \in A$

$x, y \notin A \Rightarrow x \sim y$ iff $x = y$

(cont.) cells of X/A

- 0-cells : 0-cells of $X \setminus A$ and
  one new 0-cell $[A]$
- n-cells for $n>0$:
  n-cells of $X \setminus A$

$$e^n_\alpha = \Phi^n_\alpha(\mathring{D}^n) \not\subset A \ (\Leftrightarrow e^n_\alpha \subset X \setminus A)$$

$$\phi'_\alpha : S^{n-1} \xrightarrow[\phi_\alpha]{} X^{n-1} \ (\subset X) \to \underbrace{(X/A)^{n-1}}_{\substack{\text{union of } m\text{-cells} \\ \text{for } m<n}}$$

$(X/A)^{n-1}$: $(n-1)$-skeleton of X/A

$$\left((X/A)^k = (X/A)^{k-1} \amalg \left(\coprod_\beta D^k\right)/\sim \right)$$

claim $(X/A)^k$ = img of X^k under $X \to X/A$