More basic operations

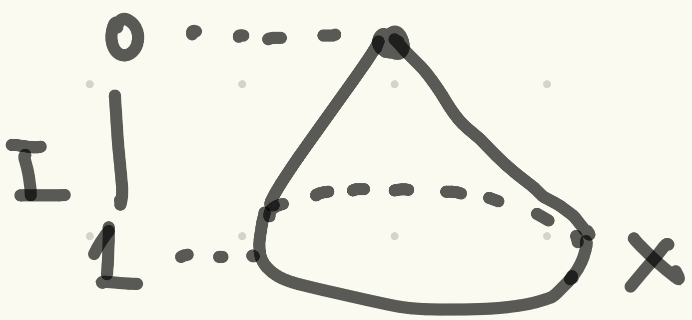
$I = [0, 1]$: closed interval

X : topological space

(without cell cplx structure for now)

Cone over X : $(\mathrm{Con}(X), CX)$

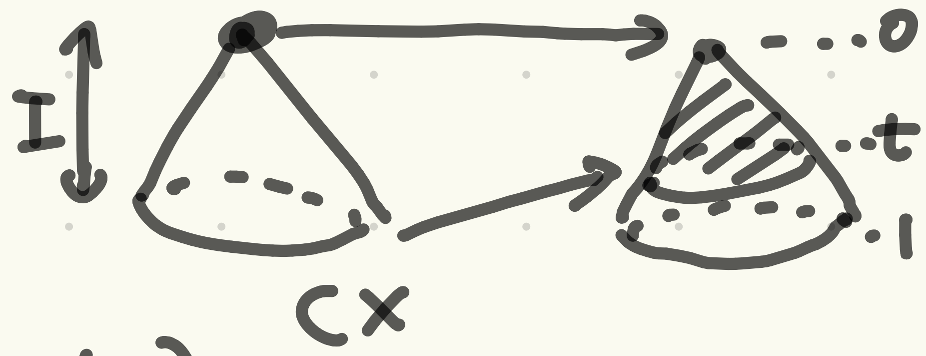
$$CX = (X \times I) / \underbrace{X \times \{0\}}_{\text{collapse this to a pt}}$$



$\{[X \times \{0\}]\} \subset CX$ def. retr.

Concrete homotopy $\mathrm{id}_{CX} \underset{\text{homotopy}}{\simeq} ir$ for

$r : CX \to \{[X \times \{0\}]\} = pt$ const. map



$F : CX \times I \to CX$

$$F(\underbrace{(\overset{X}{x}, \overset{I}{s})}_{\text{img in } CX}, \overset{I}{t}) = \underbrace{(x, st)}_{\text{img in } CX}$$

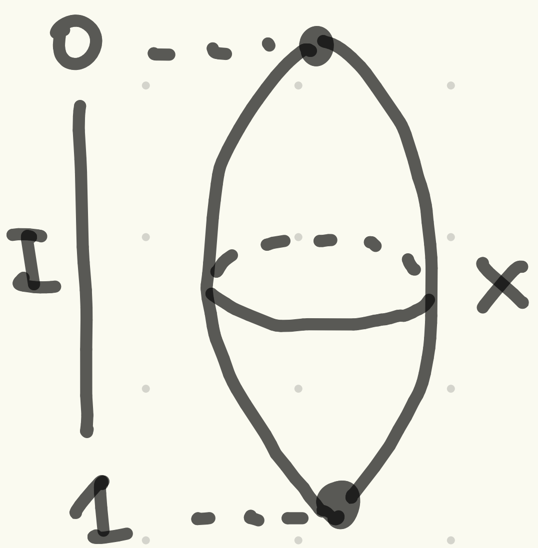
$$F((x,s),0) = (x,0) = ir(x,s)$$

$$F((x,s),1) = (x,s) = \mathrm{id}_{CX}(x,s)$$

Suspension over X : $SX = CX/(X\times\{1\})$

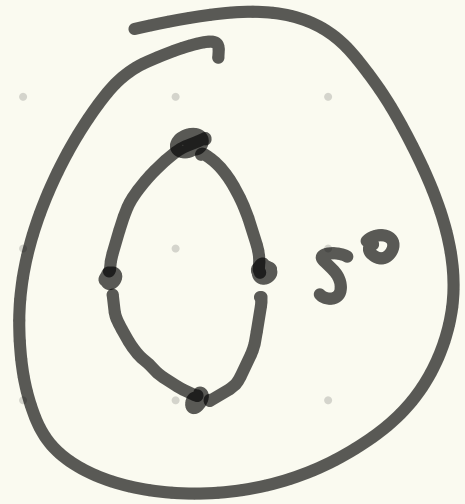
collapse $X\times\{0\}\subset X\times I$ to a point

$X\times\{1\}\subset X\times I$ to another



Ex. $SS^1 \cong S^2$

$$S\underbrace{S^0}_{\text{2-pts}} \cong S^1$$

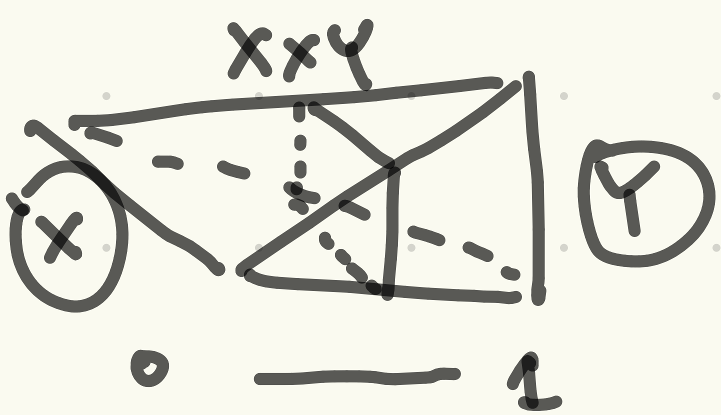


$$SS^n \cong S^{n+1}$$

Join of two spaces: $X * Y$

$$X * Y = (X \times Y \times I) / \text{equiv rel} \sim$$

- $(x, y_1, 0) \sim (x, y_2, 0)$ for any $x \in X$, $y_1, y_2 \in Y$
- $(x_1, y, 1) \sim (x_2, y, 1)$ for any $x_1, x_2 \in X$, $y \in Y$



- $(x_1, y_1, t) \sim (x_2, y_2, s)$
  $x_1 = x_2,\ y_1 = y_2,\ 0 < t = s < 1$

Alternative def.

$X * Y \overset{\sim}{=}$ "convex combinations"

of $x \in X$ and $y \in Y$

$(X \times Y \times I)/I$

$(x, y, t) \longleftrightarrow (1-t)x + ty$

$1x + 0y = x$

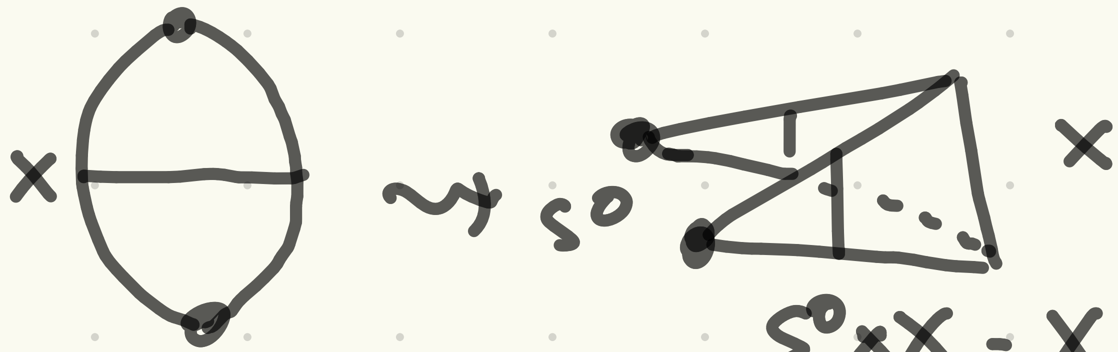
$0x + 1y = y$

If $X \subset \mathbb{R}^m$, $Y \subset \mathbb{R}^n$

$\rightsquigarrow X * Y \subset \mathbb{R}^{m+n}$ by above corr.

Suspension as join

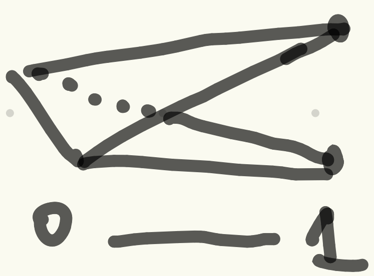
Prop. $SX \cong S^0 * X$



$S^0 \times X = X \amalg X$

Sketch : $SX \cong$ quot. of $[-1,1] \times X$

$$\begin{array}{ccc} & & \begin{array}{c} \rotatebox{90}{$\in$} \\ (t, x) \\ \downarrow \\ \end{array} \\ S^0 * X \cong X * \overset{\{\pm 1\}}{\overset{\shortparallel}{S^0}} & \ni & (x, \mathrm{sign}(t), |t|) \end{array}$$



Wedge sum of two pointed spaces

pointed space: (X, x_0) (often X)

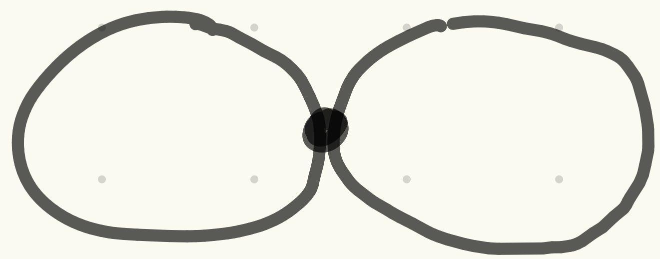
space X and a point $x_0 \in X$
"base point"

$$X \vee Y = (X \amalg Y)/x_0 \sim y_0$$

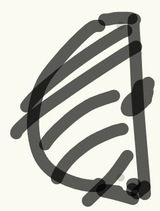
[depends on x_0, y_0 in general]

Rem $X \vee Y = (X \times \{y_0\}) \cup (\{x_0\} \times Y) \subset X \times Y$

Often: the choice of $x_0 \in X$, $y_0 \in Y$ does not matter up to homotopy

$S^1 * S^1$: 

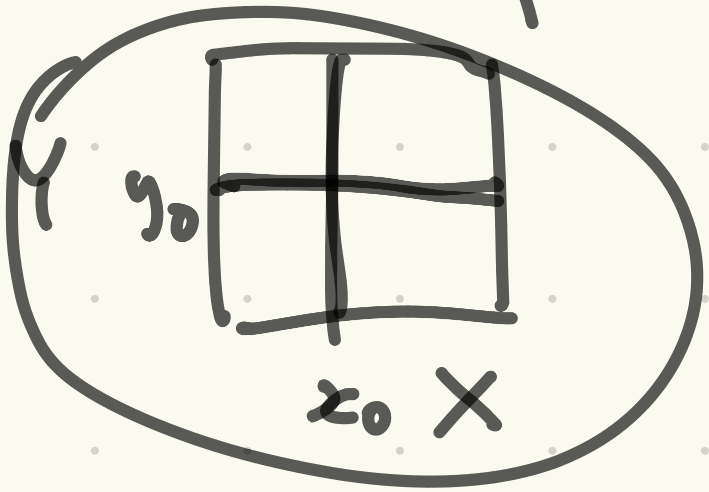
$D^2 * D^2$:  $\simeq$ 

via  $\rightsquigarrow$ 

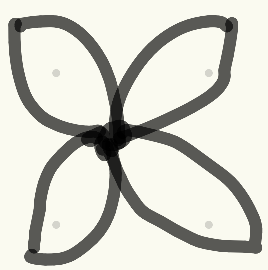
Smash product of pointed spaces

$x_0 \in X$, $y_0 \in Y$ fixed

$$\underbrace{X \wedge Y}_{\text{generally depends on choice of basepoints}} = (X \times Y) \Big/ \underbrace{\overbrace{(X \times \{y_0\}) \cup (\{x_0\} \times Y)}^{\cong X \vee Y}}_{\text{collapsed to a point}}$$



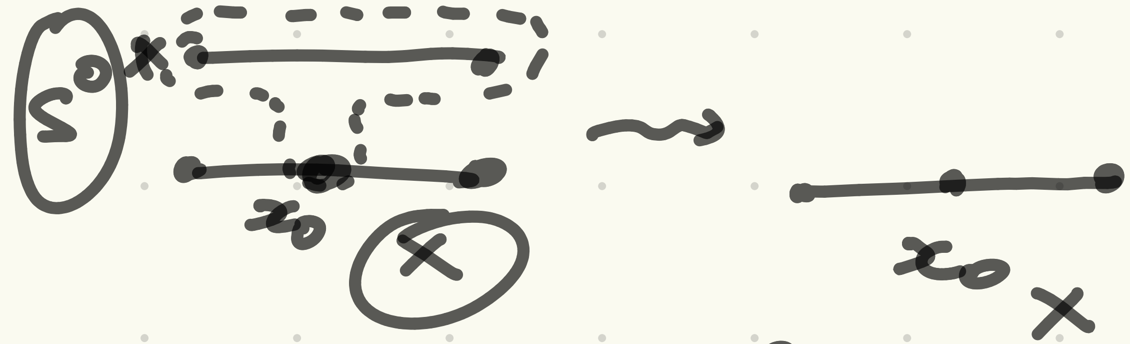
$\rightsquigarrow$



$X \wedge Y$

Examples

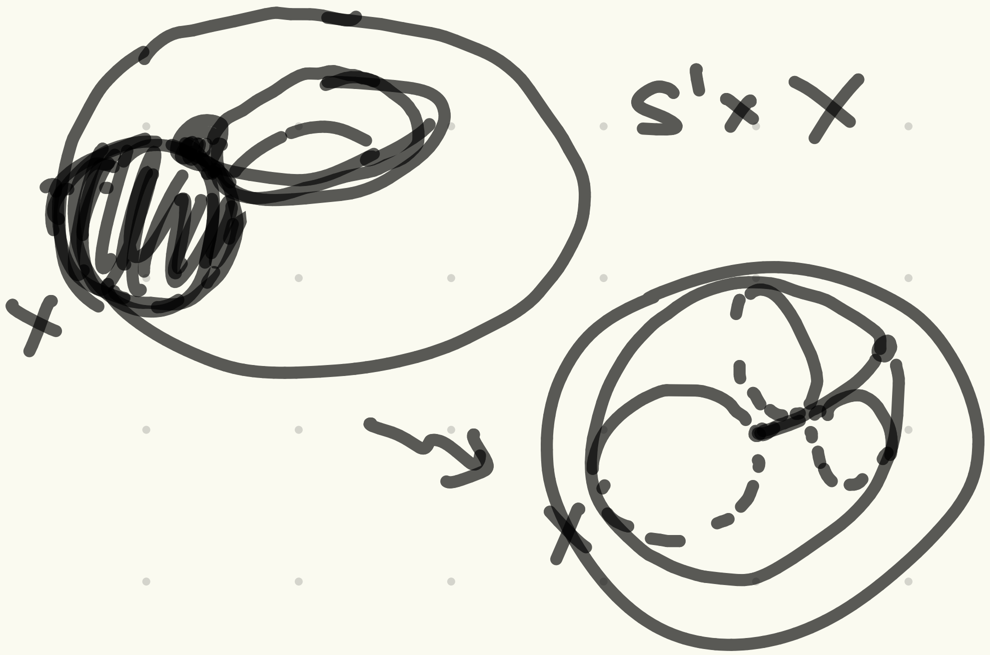
- $S^0 \wedge X \cong X$



if X is nice

- $S^1 \wedge X \cong SX / \{x_0\} \times I \simeq SX$

homotop equiv.



Exercise $S^m \wedge S^n \underline{\underline{\simeq}} S^{m+n}$

(cf. below for cell cplx structure)

Rem The above constructions

CX, SX, $X * Y$, $X \vee Y$, $X \wedge Y$

are <u>functorial</u> in X and Y

i.e. $f : X \to Z$ continuous map

$\rightsquigarrow$ $SX \xrightarrow{Sf} SZ$ induced cont map (cont.) $\Sigma g\,\Sigma f = \Sigma gf$ for

$$X \xrightarrow{f} Z \xrightarrow{g} W, \text{ etc.}$$

$$\rightsquigarrow \Sigma X \to \Sigma Z \to \Sigma W$$

<u>Mapping</u> <u>cone</u> of $f : X \to Y$

$$Cf = \underbrace{CX}_{X \times I / X \times \{0\}} \amalg Y \Big/ (x,1) \sim f(x) \quad (x \in X)$$

Compatibility with cell cplx structure

$CX, SX, X * Y, X \vee Y, X \wedge Y$ are

built up using these ops:

- take $X \times Y$ (with $Y = I$)
- collapse $X \times \{*\}$ for $* = 0, 1, y_0, \dots$

$\rightsquigarrow$ will carry over cell cplx structure

from X & Y

I : cell cplx with

- two 0-cells $\{0\}, \{1\}$
- one 1-cell $e^1 = (0, 1)$

i.e. $I^0 = \{0, 1\}, \; I^1 = I^2 = \cdots = I$

X cell cplx $\rightsquigarrow$ $X \times I$ product cell cplx

$X \times \{0\}, X \times \{1\} \subset X \times I$ subcplx

$$\rightsquigarrow \quad CX = (X \times I)/(X \times \{0\})$$

is the quotient of the CW pair

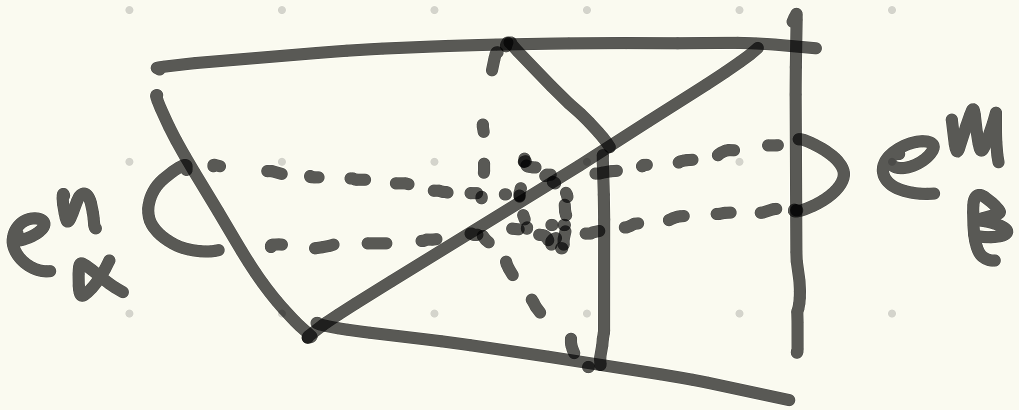
$$(X \times I, X \times \{0\}) \rightsquigarrow \text{cell cplx str. on } CX$$

SX : cell cplx as $CX/X \times \{1\}$

$$CX : \underbrace{[X \times \{0\}]}_{0\text{-cell}}, \quad \underbrace{\underbrace{e^n_\alpha}_{\substack{n\text{-cell} \\ \text{in } X}} \times \underbrace{(0,1)}_{\substack{1\text{-cell} \\ \text{of } I}}}_{(n+1)\text{-cell of } CX}, \quad \underbrace{e^n_\alpha \times \{1\}}_{\substack{n\text{-cell} \\ \text{of}}}$$

$X * Y$: cell cplx with cells

$$\underbrace{e^n_\alpha \times pt \times \{0\}}_{\text{from } X}, \quad \underbrace{pt \times e^m_\beta \times \{1\}}_{\text{from } Y}, \quad \underbrace{e^n_\alpha \times e^m_\beta \times (0,1)}$$



pointed cell complex $x_0 \in X^0 \subset X$

$$\rightsquigarrow \quad \underbrace{\{x_0\} \times Y}, \quad \underbrace{X \times \{y_0\}} \subset X \times Y$$

subcomplexes

$X \vee Y$: cell cplx as $(\{x_0\} \times Y) \cup (X \times \{y_0\})$

$X \wedge Y$: quot. cplx

$$(X \times Y) / \left((\{x_0\} \times Y) \cup (X \times \{y_0\}) \right)$$

S^m has cells $\left(\begin{array}{l} \text{one 0-cell } e^0 \\ \text{one } m\text{-cell } e^m \end{array} \right.$

$\rightsquigarrow S^m \wedge S^n$ will have

$e^0 \times e^0$, , $e^m \times e^n$

0-cell $\quad$ $(m+n)$-cell

$\left(\begin{array}{l} \text{same cell} \\ \text{cplx structure} \\ \text{as } S^{m+n} \end{array} \right.$

How to produce homotopy equivalence

(without formal proofs for now)

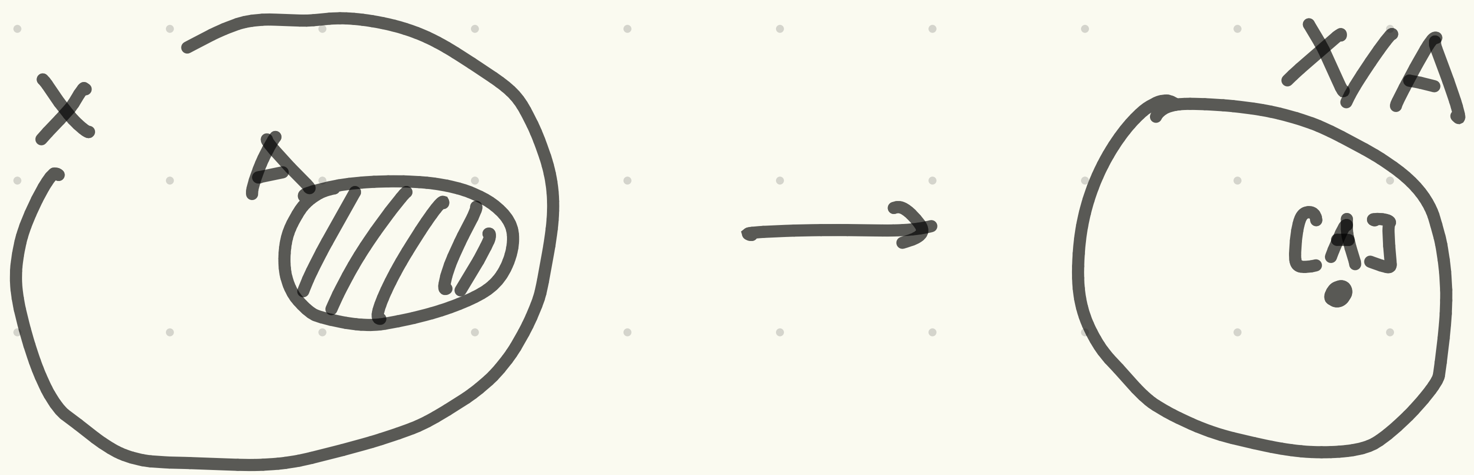
- collapse contractible subspace

  X : cell complex

  A : subcomplex, <u>contractible</u>

  ($A \simeq pt$)

  then the canonical map $X \to X/A$ is a homotopy equiv.



Rem. we still need to give a "homotopy inverse" $g : X/A \to X$

Glueing over a subspace

X_0, X_1 : top. spaces

$A \subset X_1$: subspace

$f : A \to X_0$: cont. map

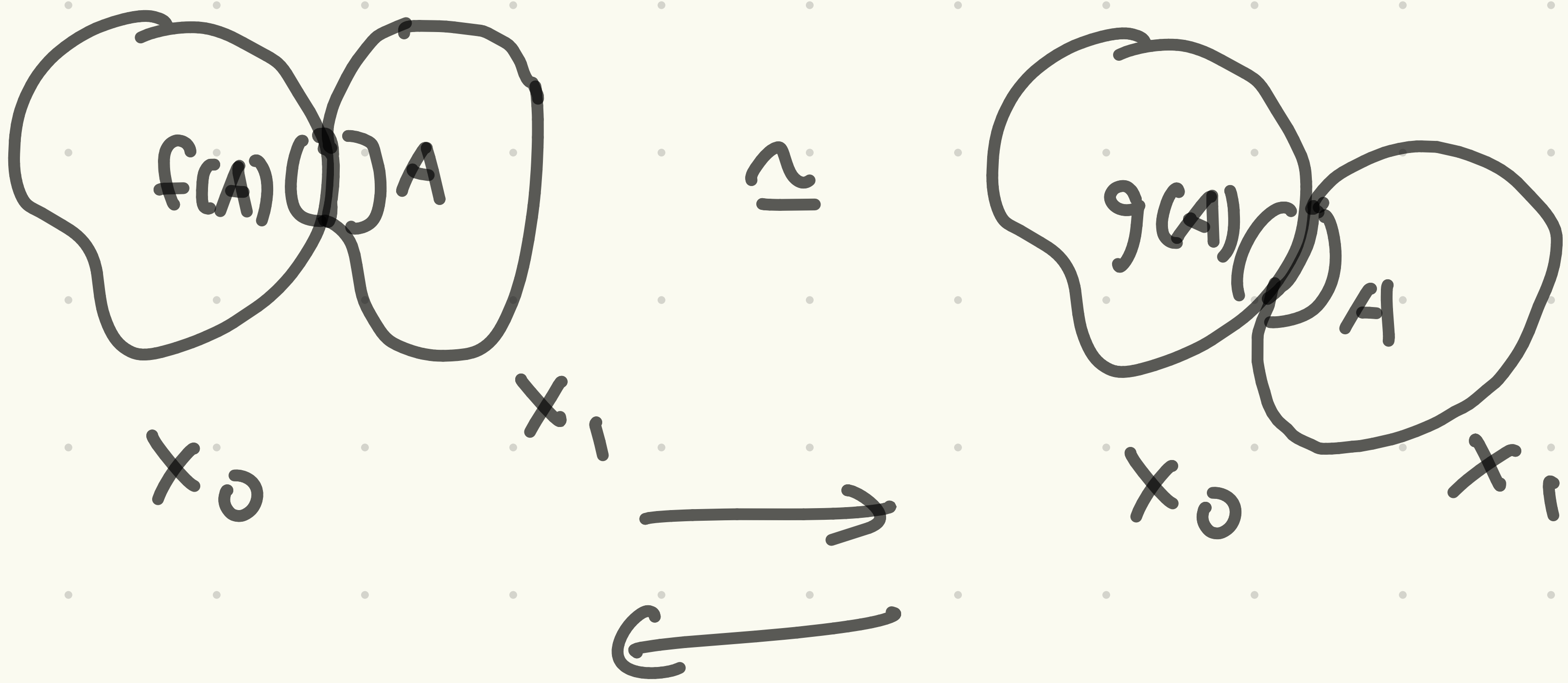
$\rightsquigarrow \quad X_0 \sqcup_f X_1 = X_0 \amalg X_1 / f(a) \sim a \ (a \in A)$

$$\underset{f(a)}{\overset{X_0}{\cup\!\!\!\!\!\!\;\;\;\;\!\!\;\;\,\,}} \qquad \underset{A \ni a}{\overset{X_1}{\cup}}$$

Suppose

- (X_1, A) is a CW pair
- $f, g : A \to X_0$ homotopic

then $X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1$



f(A)
A
X0
X1
≃
g(A)
A
X0
X1