Homotopy Extension Property

X : topological space

$A \subset X$, suppose we have a deformation data on A

(deforming A to another space, homotopy of cont. maps to Y, ...)

$\rightsquigarrow$ when can we "extend" this data to X ?

Motivation: if A is contractible, does $A \simeq pt$ extend to $X \simeq X/A$?

Def. subspace $A \subset X$ has the (also say (X,A) has HEP) <u>homotopy</u> <u>extension</u> <u>property</u> if,

$\forall\ Y,\ f_0 : X \to Y,\ F : A \times I \to Y$ s.t. ($I = [0,1]$)

$F(a,0) = f_0(a)$ for $a \in A$

$\exists\ \tilde{F} : X \times I \longrightarrow Y$ s.t. $\tilde{F}(a,t) = F(a,t)$ for $a \in A,\ t \in I$

$f_0(x) = \tilde{F}(x,0)$ for $x \in X$

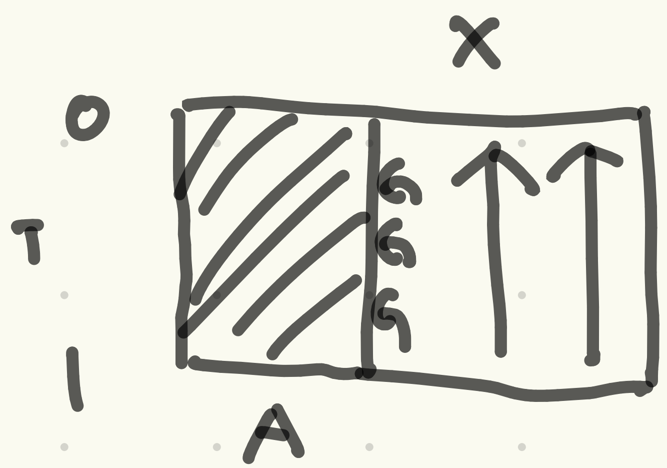
Prop. $A \subset X$ has HEP

$\iff (X \times \{0\}) \cup (A \times I) \subset X \times I$ is a deformation retract

i.e. $\exists\ r : X \times I \to (X \times \{0\}) \cup (A \times I)$

s.t. $\mathrm{id}_{X \times I} \simeq i r$, $\quad r i = \mathrm{id}_{X \times \{0\} \cup A \times I}$

$\quad\quad\quad\quad\quad\quad\quad\quad\uparrow$ incl,



Proof $\Rightarrow$: take $Y = (X \times \{0\}) \cup (A \times I)$

$f_0 : X \to X \times \{0\}$, $x \mapsto (x, 0)$

$F = \text{"}id_{A \times I}\text{"} : A \times I \to Y$

$\rightsquigarrow \tilde{F} : X \times I \to Y$ does the job as r

$\Leftarrow$ when A is closed (automatic if X is Hausdorff)

$f_0 : X \to Y$ and $F : A \times I \to Y$ agree on A "=" $A \times \{0\}$

X, $A \times I \subset (X \times \{0\}) \cup (A \times I)$ closed
as $X \times \{0\}$

$\Rightarrow$ get $g : (X \times \{0\}) \cup (A \times I) \to Y$

s.t. $g(x, 0) = f_0(x)$, $g(a, t) = F(a, t)$

Key points (for continuity of g)

- g cont $\Leftrightarrow$ g^{-1}(closed set) is closed

- $C \subset (X \times \{0\}) \cup (A \times I)$ closed

$\Leftrightarrow$ $C \cap (X \times \{0\})$, $C \cap (A \times I)$ closed (cont.) now we have

$$\begin{tikzcd}
X \times I \arrow[r, "r"] & (X \times \{0\}) \cup (A \times I) \arrow[r, "g"] & Y \\
& X \arrow[u] \arrow[ur, "f_0"] \quad A \times I \arrow[u] \arrow[ur, "F"'] &
\end{tikzcd}$$

$\tilde{F} = gr$ does the job. $\blacksquare$

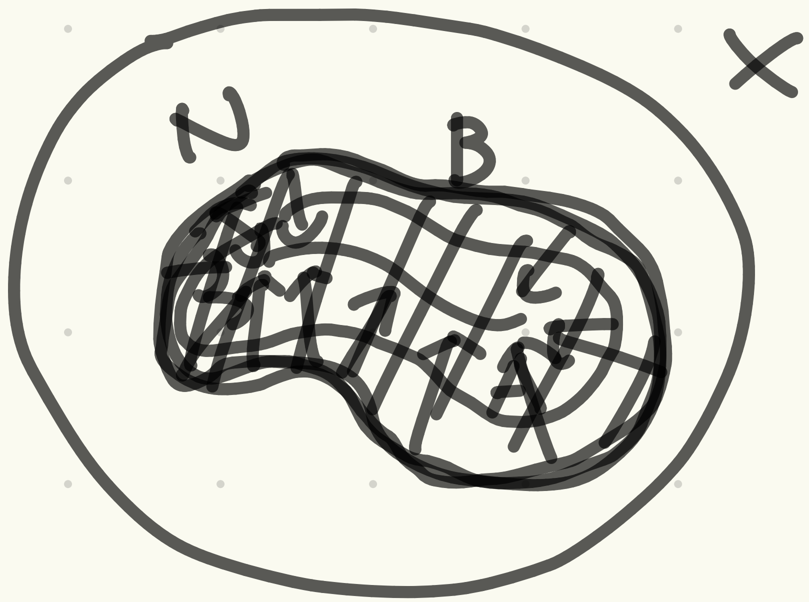
Ex. 0.15 $A \subset X$ has HEP when A has a "mapping cylinder neighborhood" N:

(closed neigh. of A in X,

$\exists\, f : B = \partial N \to A$ s.t. $M_f \cong N$ fixing ...

$\uparrow$ map. cylinder

$A \subset X$



N closed neigh. of A i.e.

$A \subset \mathring{N} \subset N$

interior

$\partial N = B$, $f : B \to A$ map

$$M_f = (B \times I \cup A) / \underset{\substack{\cap \\ B\times I}}{\underline{(b,0)}} \sim \underset{\substack{\cap \\ A}}{f(b)}$$

map cylinder

Want $M_f \cong N$ id between $B \times \{1\}$ and B, A and A

Prop 0.16 X : cell complex,

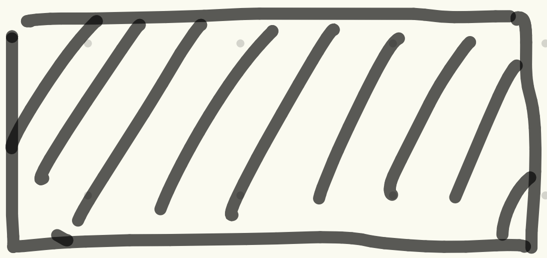
$A \subset X$ subcomplex ((X, A) CW pair)

$\Rightarrow A \subset X$ has HEP

Proof

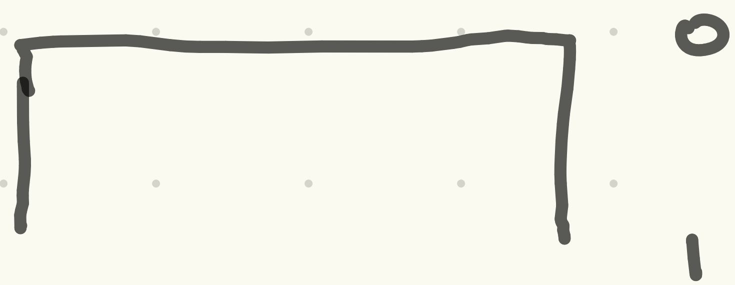
Key observation : $S^{n-1} = \partial D^n \subset D^n$

has HEP



$D^1 \times I$

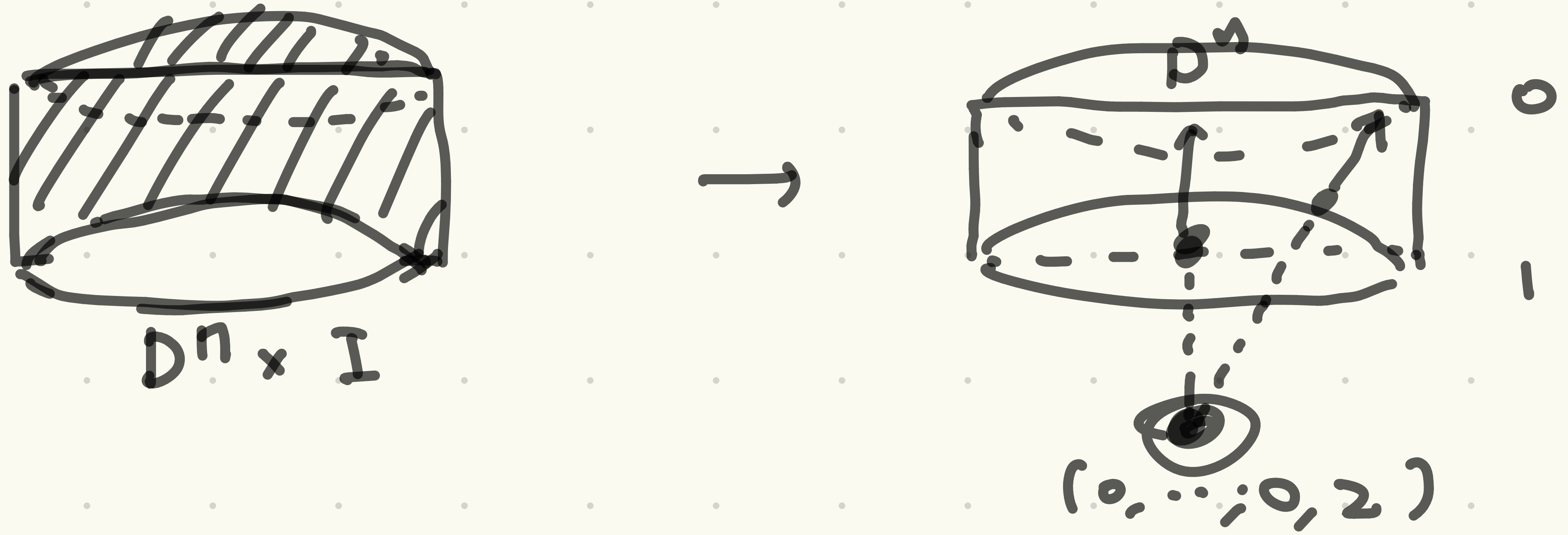
$\longrightarrow$



$(D^1 \times \{0\}) \cup (S^0 \times I)$

formally ; $D^n \times \{0\} \cup S^{n-1} \times I \subset \mathbb{R}^{n+1}$

"radial projection" from $(0, \cdots, 0, 2)$ $\in \mathbb{R}^{n+1}$



$\rightsquigarrow$ Use this to get deformation retract map for $(X \times \{0\}) \cup (A \times I) \subset X \times I$

def. retract map

$$X^n \times I \to (X^n \times \{0\}) \cup ((X^{n-1} \cup A^n) \times I) = Y_n$$

Claim:

$X^n \times I \cong Y_n \cup \left(\bigcup_{n\text{-cells of } X/A} D^n \times I \right)$ / glueing rel

glueing map:

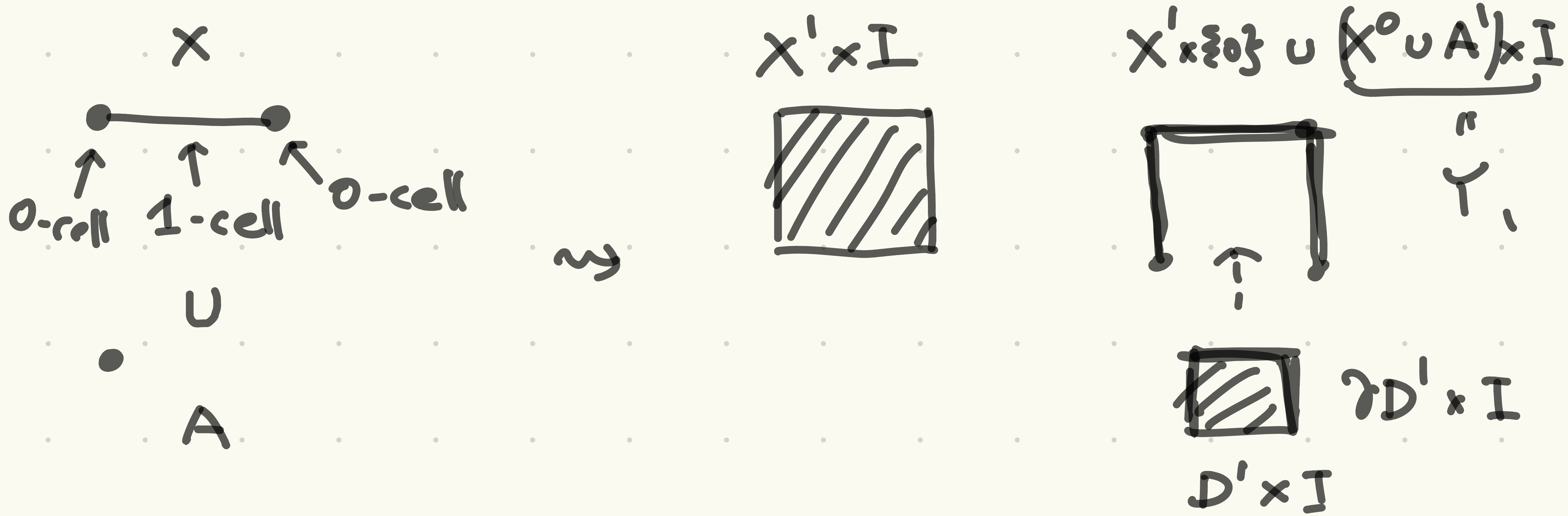
for the α-th n-cell of X

$$(D^n \times \{0\}) \cup (\partial D^n \times I)$$

$\downarrow$ $\quad$ $\downarrow$ from characteristic maps of X

$X^n \times \{0\}$ $\quad$ $X^{n-1} \times I$ $\subset Y_n$

from glueing data of X^n



We have def. retract map

$$X^n \times I \to (X^n \times \{0\}) \cup ((X^{n-1} \cup A^n) \times I) = Y_n$$

Next: "glue" these retract maps

to get $X \times I \to (X \times \{0\}) \cup (A \times I)$

On $X^n \times I$:

$$X^n \times I \xrightarrow{\text{def. ret.}} (X^n \times \{0\}) \cup ((X^{n-1} \cup A^n) \times I)$$

$$\| $$

$$X^n \times \{0\} \cup X^{n-1} \times I \cup A^n \times I$$

$$\left(\begin{array}{ccc} \downarrow \mathrm{id} & \Big\downarrow \text{def. ret.} & \downarrow \text{"id"} \\ X^n \times \{0\} & & A \times I \\ \cup & & \cup \\ \multicolumn{3}{c}{Y_{n-1} = X^{n-1} \times \{0\} \cup X^{n-2} \times I \cup A^{n-1} \times I} \end{array}\right)$$

we end up in $X^n \times \{0\} \cup X^{n-2} \times I \cup A \times I$

.. continue until we get $\ldots \cup X^0 \times I \cup A \times I$

.. then $X^n \times \{0\} \cup \underbrace{\overset{\text{disc. subset of } X}{X^0} \times I} \cup A \times I \to X^n \times \{0\} \cup A \times I$

shrink down $(X^0 \setminus A) \times I$ to $X^0 \times \{0\}$

So we get $X^n \times I \xrightarrow{\text{def. ret.}} X^n \times \{0\} \cup A \times I$

compatible with incl. $X^{n-1} \subset X^n$

$\rightsquigarrow$ collect these for all n to get

$$\underbrace{X \times I \xrightarrow{r} X \times \{0\} \cup A \times I}_{\text{def. retr.}}$$

Rem. collecting homotopy

We have: def. retract

$$X^n \times I \xrightarrow{r^{(n)}} (X^n \times \{0\}) \cup ((X^{n-1} \cup A^n) \times I)$$

$\mathrm{Id}_{X^n \times I} \simeq i r^{(n)}$ i.e. family of

maps $f_t^{(n)} : X^n \times I \to X^n \times I$ ...

For $X \times I \xrightarrow{r} X \times \{0\} \cup A \times I$:

define $\tilde{f}_t$ connecting $\mathrm{id}_{X \times I}$ and ir

by: if $x \in X^n$, then

$$\tilde{f}_t(\underbrace{x,s}_{\substack{\cap \\ X^n \times I \subset X \times I}}) = \begin{cases} (x,s) & t < \frac{1}{2^{n+1}} \\ \text{reparametrized } f_t^{(n)}(x,s) & \frac{1}{2^{n+1}} \le t \le \frac{1}{2^n} \\ \text{reparametrized } f_t^{(n-1)}\left(f_1^{(n)}(x,s)\right) & \frac{1}{2^n} \le t \le \frac{1}{2^{n-1}} \\ \cdots \end{cases}$$

connects (x,s) to $r^{(n)}(x,s)$

$r^{(n)}(x,s)$

to

$r^{(n-1)} r^{(n)}(x,s)$

$\rightsquigarrow r^{(0)} \cdots r^{(n)}(x,s) = r(x,s)$

Applications: homotopy equivalence relations for operations

- $X \rightsquigarrow X/A$ for contractible $A \subset X$

- $\underbrace{X_0 \sqcup_{f_t} X_1}$ for $f_t : A \to X_0$, $A \subset X_1$

Prop 0.17 X : cell complex

$A \subset X$ subcomplex, contractible

$\Rightarrow$ natural map $X \xrightarrow{p} X/A$ is a homotopy equivalence

Proof $A \subset X$ has HEP

Take $f_0 = id_X : X \to X = Y$

$F : A \times I \to A \subset X$ homotopy between id_A and $A \to pt \to A \subset Y$

$X \overset{\shortparallel}{} Y$

img $a_0 \in A$

We get $\tilde{\hat{F}} : X \times I \to X^{(=Y)}$ extending

$f_0 = id_X$ and F, $\tilde{\hat{F}}(x,0) = id_X(x) = x$

$\tilde{\hat{f}}_1(x) = \tilde{\hat{F}}(x,1)$ sends A to $\{a_0\}$

$\rightsquigarrow \tilde{\hat{f}}_1$ is $X \xrightarrow[\text{nat. map}]{p} X/A \xrightarrow[g \text{ represented by } f_0]{} X$

Claim: g is a homotopy inverse of p ($X \simeq X/A$)

$\because \tilde{\hat{F}}$ implements $id_X \simeq g\,p$. $\square$

Prop. 0.18 X_1 cell complex

$A \subset X_1$ subcomplex

X_0 : top. space

$f, g : A \to X_0$ cont. maps $\underset{\text{homotop.}}{f \simeq g}$

$\Rightarrow X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1$ ($= X_0 \amalg X_1 / a \sim g(a)$ for $a \in A$)

Proof $F : A \times I \to X_0$ homotopy between f and g

$\rightsquigarrow$ take $X_0 \sqcup_F (X_1 \times I)$

Key observation:

Def. retr. $X_1 \times \{0\} \cup A \times I \subset X_1 \times I$

induces def. retr. $X_0 \sqcup_f X_1 \subset X_0 \sqcup_F (X_1 \times I)$

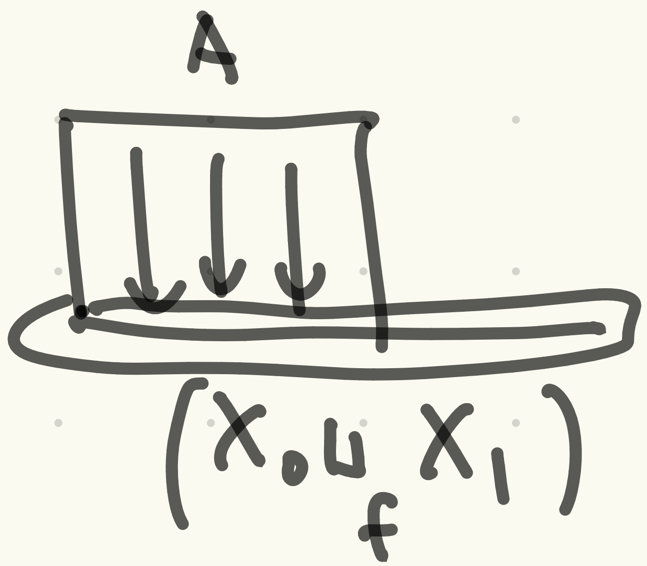
$\uparrow$ at $X_1 \times \{0\}$

$\overset{\cup}{A \times \{0\}}$

$$X_0 \sqcup_F (X_1 \times I) \xrightarrow{\text{retr}} X_0 \sqcup_F (X_1 \times \{0\} \cup A \times I)$$

$$\| $$

$A \times I$ glued to $X_0 \sqcup_f X_1$



i.e. map. cylinder of

$$A \xrightarrow[F]{} X_0 \xrightarrow[\text{incl}]{} X_0 \sqcup_f X_1$$

Similarly we get def. retract

$X_0 \sqcup_g X_1 \subset X_0 \sqcup_F (X_1 \times I)$ so

$\uparrow$ at $X_1 \times \{1\}$

$$X_0 \cup_f X_1 \simeq X_0 \sqcup_F (X_1 \times I) \simeq X_0 \sqcup_g X_1$$