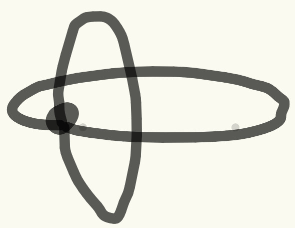
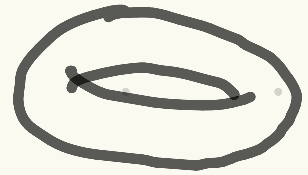
Exercise session 1

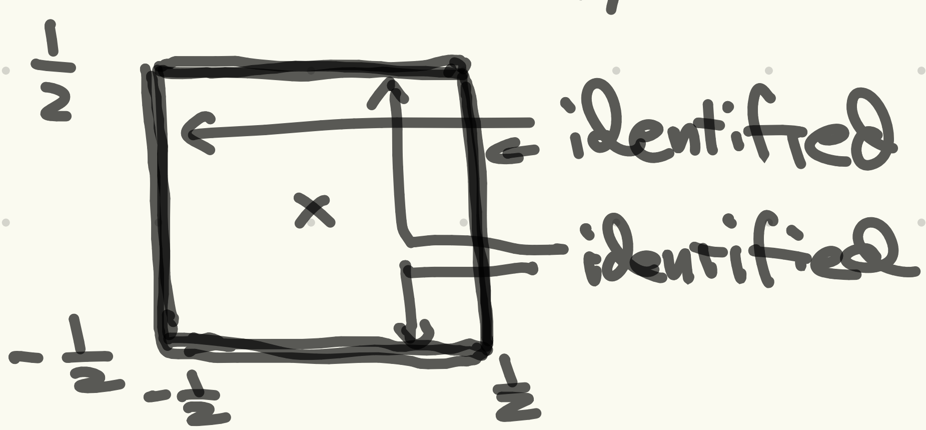
Problem 1 explicit presentation of deform. retract

for  $\subset$  $\setminus \{pt\}$ (punctured torus)

$S^1 \vee S^1$ torus $S^1 \times S^1$

We take $(\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$ as a model of torus T

- remove the point $p = [(0,0)]$ (image of $(0,0)$)
- look at $A = \{[(x,y)] : x = \frac{1}{2} \text{ or } y = \frac{1}{2}\}$
  as the copy of $S^1 \vee S^1$ to retract to

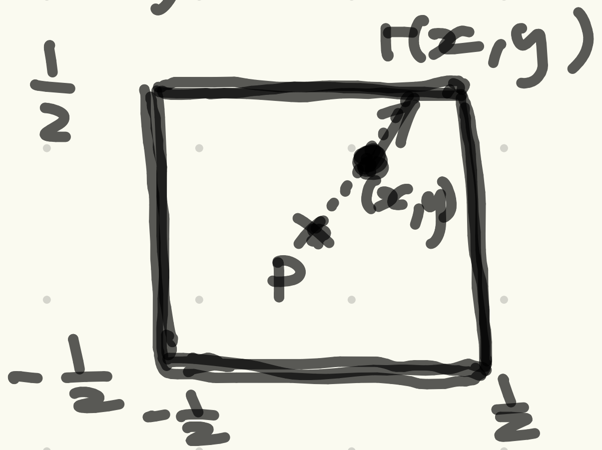


A: boundary

the map $r : T \setminus \{p\} \to A$

- characterization : in $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$,

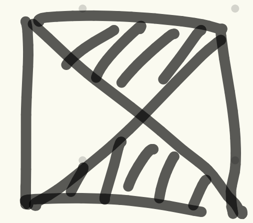
  (x,y) lies on the ray from $(0,0)$ to $r([x,y])$



"a lift on the boundary of $[-\frac{1}{2}, \frac{1}{2}]^2$ of"

- concrete formula

  when $|y| \geq |x|$ $\quad r([x,y]) = [\frac{1}{2|y|}(x,y)]$



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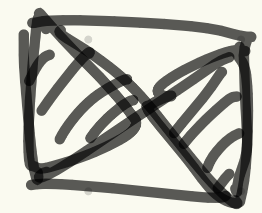
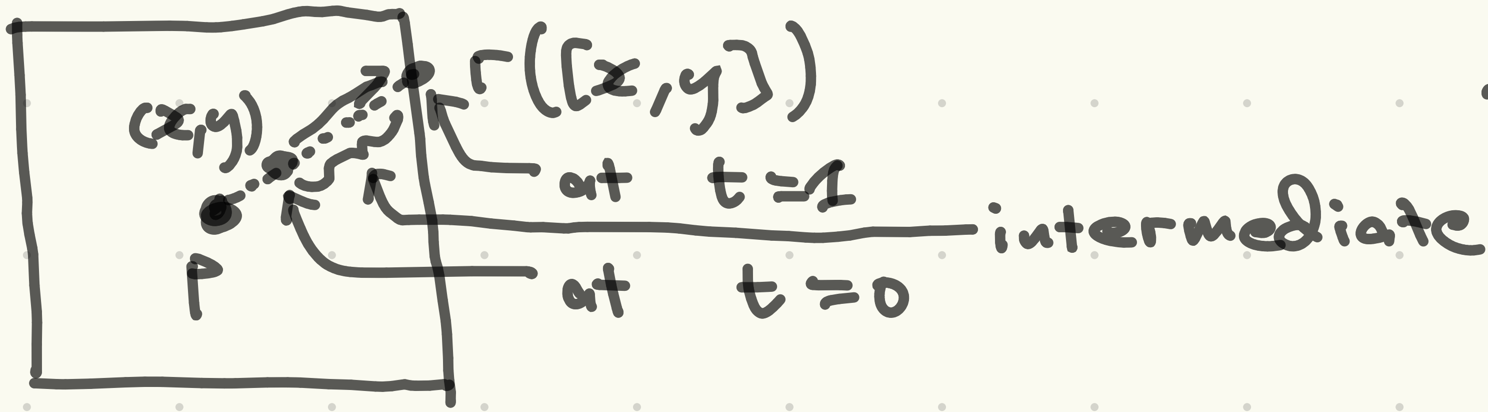


$r i = id_A$

homotopy between $\mathrm{id}_{T\setminus\{p\}}$ and $i\,r$

i: inclusion map

idea: move $[(x,y)]$ to $r([(x,y)])$ on the ray between p and $r([(x,y)])$



we can take convex combination

concrete formula: $F : (T\setminus\{p\})\times I \to T\setminus\{p\}$

$$F([x,y],t) = \begin{cases} \left[\left((1-t)+\frac{t}{2|y|}\right)(x,y)\right] & |y|\geq|x| \\ \left[\left((1-t)+\frac{t}{2|x|}\right)(x,y)\right] & |x|\geq|y| \end{cases}$$

Problem 11

$f : X \to Y$ is a <u>homotopy equivalence</u>

$\hookrightarrow$ admits "homotopy inverse"

$(\tilde{g} : Y \to X$ s.t. $f\tilde{g} \simeq id_Y$, $\tilde{g}f \simeq id_X)$

$\Leftrightarrow$ $\exists\, g, h : Y \to X$ s.t. $fg \simeq id_Y$, $hf \simeq id_X$

$\Leftarrow$

Key observation: $hfg : Y \to X$ is homotopy equivalent to both g and h

$(hf)g \simeq id_X g = g$, $h(fg) \simeq h\, id_Y = h$

$\rightsquigarrow$ $gf \simeq hfgf \simeq hf \simeq id_X$

concretely : if $F: X \times I \to X$ is s.t.

$F(x,0) = x$, $F(x,1) = hf(x)$

then $G: Y \times I \to X$, $(y,t) \mapsto F(g(y),t)$

satisfies $G(y,0) = g(y)$, $G(y,1) = hfg(y)$

i.e. F gives homotopy $id_X \simeq hf$

$\Rightarrow$ $F \circ (g \times id_I)$ gives homotopy $g \simeq hfg$

To get $gf \simeq id_X$

$$\underset{F \circ (gf \times id_I)}{gf \simeq hfgf} \quad \underset{h \circ F' \circ (f \times id_I)}{\simeq hf} \quad \underset{F}{\simeq id_X}$$

with $F': Y \times I \to Y$ giving $fg \simeq id_Y$

$F'(y,0) = fg(y)$, $F'(y,1) = y$

$$[g] = [hfg] = [h] = [f]^{-1}$$

Generalization

fg, hf homotopy equivalences

$\Rightarrow$ f homotopy equivalence

$fgg' \simeq \mathrm{id}_Y$ for $g' : Y \to Y$

$h'hf \simeq \mathrm{id}_X$ for $h' : X \to X$ as before

$g'' = gg'$, $h'' = h'h$ satisfy same assumption

Problem 16 $\quad S^\infty = \bigcup_{n=0}^{\infty} S^n$ is contractible

- S^n is embedded in the "equator" of S^{n+1}

i.e. $S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}, \ \sum x_i^2 = 1\}$

$$\downarrow$$

$$(x_1, \dots, x_{n+1}, 0) \in S^{n+1} \subset \mathbb{R}^{n+2}$$

- topology of S^∞: as cell complex

$U \subset S^\infty$ is open $\Leftrightarrow \forall n \ \ U \cap S^n \subset S^n$ is open

$\rightsquigarrow f : S^\infty \to X$ is continuous

$\Leftrightarrow \forall n \ \ f|_{S^n} : S^n \to X$ is cont.

$F : S^\infty \times I \to X$ is continuous

$\Leftrightarrow \forall n \ \ F|_{S^n \times I} : S^n \times I \to X$ is cont.

Key idea:

- $S^\infty = \{ (x_1, x_2, \dots) : x_i \in \mathbb{R},\ x_i = 0$ except for finitely many i, $\sum x_i^2 = 1 \}$

- points of the form $(0, x_2, x_3, \dots)$ can be connected to $(1, 0, 0, \dots)$ by

$$\left(\sin t\tfrac{\pi}{2}, \cos t\tfrac{\pi}{2} \cdot x_2, \cos t\tfrac{\pi}{2} \cdot x_3, \dots\right) \quad 0 \le t \le 1$$

$\rightsquigarrow$ we want the map $\underline{(x_1, x_2, \dots) \mapsto (0, x_1, x_2, \dots)}$ to be homotopic to id_{S^∞}

homotopy between id_{S^∞} and the "shift" map

$f : S^\infty \to S^\infty$, $(x_1, x_2, \ldots) \mapsto (0, x_1, x_2, \ldots)$

idea: - make sure that at time 2^{-k} the $(k+1)$-th coordinate is 0

- between time $\underbrace{2^{-k} \text{ and } 2^{-k+1}}$ move the k-th coordinate to the $(k+1)$-th

$\underbrace{(\ldots, x_k, 0, x_{k+2})} \rightsquigarrow \underbrace{(\ldots, 0, x_k, x_{k+2}, \ldots)}$

by $(\ldots, \underbrace{\cos\left(2^k(t - 2^{-k})\tfrac{\pi}{2}\right) x_k}_{\text{move from } 0 \text{ to } 1}, \underbrace{\sin\left(2^k(t-2^{-k})\tfrac{\pi}{2}\right) x_k}, \ldots)$

Formal definiton of homotopy

$$F : S^\infty \times I \to S^\infty, \quad (x_1, \cdots) \mapsto (x_1^{(t)}, x_2^{(t)}, \cdots)$$

$$x_k^{(t)} = \begin{cases} x_k \quad (t \le 2^{-k}), \quad x_{k-1} \quad (t \ge 2^{-k+2}) \\ \cos\left(2^k(t-2^{-k})\frac{\pi}{2}\right)\cdot x_k \quad (2^{-k} \le t \le 2^{-k+1}) \\ \sin\left(2^{k-1}(t-2^{-k+1})\frac{\pi}{2}\right)\cdot x_{k-1} \quad (2^{-k+1} \le t \le 2^{-k+2}) \end{cases}$$

so $F((x_1, \cdots), 0) = (x_1, x_2, \cdots)$

$F((x_1, \cdots), 1) = (0, x_1, x_2, \cdots)$

$$F|_{S^n \times I}((x_1, \cdots, x_{n+1}, 0, \cdots), t)$$

$$= \begin{cases} (x_1, \cdots, x_{n+1}, 0, \cdots) & \text{for } t \le 2^{-(n+1)} \\ \text{continuous formula} & \text{for } t \ge 2^{-(n+1)} \end{cases}$$

$\rightsquigarrow$ continuous as $S^n \times I \to S^\infty$

So we get $\mathrm{id}_{S^\infty} \underset{F}{\simeq} \underset{\text{shift}}{f} \simeq (S^\infty \to \{(1,0,\cdots)\})$

$(\sin t, \cos t \cdot x_1, \cdots)$

i.e. $\underline{\{(1,0,\cdots)\}} \subset S^\infty$ is a deform. retract

$f : S^\infty \to S^\infty$ is homotopy equiv. to

$S^\infty \to \{(1,0,\cdots)\}$

$G((x_1,\cdots),t) = (\sin t\frac{\pi}{2}, \cos t\frac{\pi}{2}\cdot x_1, \cos t\frac{\pi}{2}\cdot x_2 \cdots$

$G((x_1,\cdots),0) = (0, x_1, \cdots) = f(x_1, x_2, \cdots)$

$G((x_1,\cdots),1) = (1,0,\cdots)$

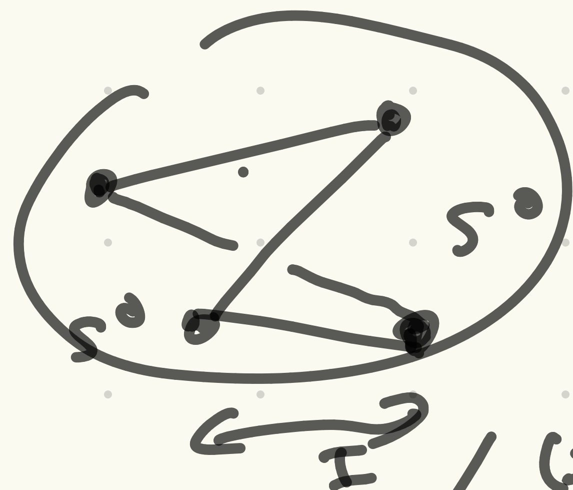
Problem 18 $\quad S^m * S^n = S^{m+n+1}$

Key idea:

- $X_1 * \cdots * X_n$ can be interpreted as the space of convex combinations

$$\underline{t_1 x_1 + \cdots + t_n x_n}, \quad 0 \le t_i, \ \sum t_i = 1, \ x_i \in X_i$$

- $S^1 \cong S^0 * S^0$



$S^0 \subset \mathbb{R}$ as $\{\pm 1\}$

$\mathbb{R} \times \mathbb{R} \times I \subset \mathbb{R}^3$

$$\hookrightarrow S^0 \times S^0 \times I \Big/ \begin{array}{l} (x,y,0) \sim (x,y',0) \\ (x,y,1) \sim (x',y,1) \end{array}$$

$S^m * S^n \subset \mathbb{R}^{m+1+n+1+1}$

as $\{((1-t)x, ty, t) : 0 \le t \le 1$

$x \in S^m \subset \mathbb{R}^{m+1}, \ y \in S^n \subset \mathbb{R}^{n+1}\}$

conv. combination of $\underbrace{(1-t)x+ty}$ and $\underbrace{z}$

$\rightsquigarrow \underline{(1-s)((1-t)x+ty)+sz}$

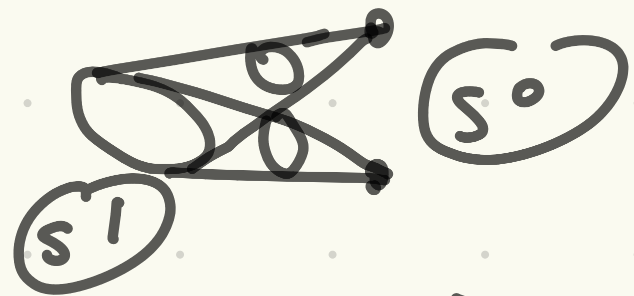
convex comb. of x, y, z

i.e. $\underline{t_1 x + t_2 y + t_3 z}$

$0 \le t_i, \ \sum t_i = 1$

$S^1 * S^1 \cong S^3$

$S^1 \cong S^0 * S^0 \quad \Rightarrow \quad S^1 * S^1 = \underbrace{\underbrace{S^0 * S^0}_{S^1} * S^0}_{S^2} * S^0$



glue two D^2 along their boundaries

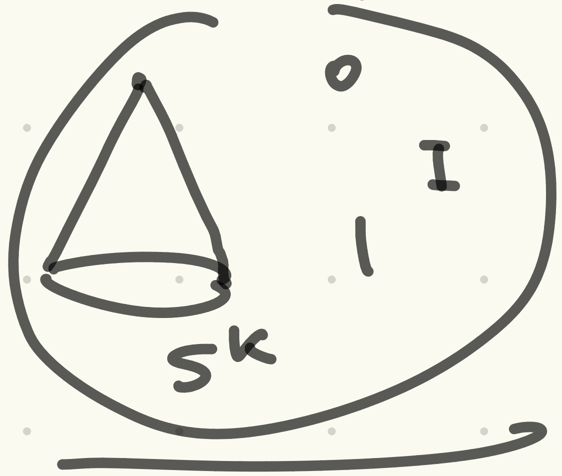
$S^2 * S^0 \cong S^3$ again as glueing of $D^3 \cong \underset{\text{cone}}{C}\, S^2$ along their boundaries

$S^m * S^n \cong S^{m+n+1}$

similarly $S^m * S^n \cong \underbrace{S^0 * \cdots * S^0}_{m+1+n+1 \text{ copies}}$ $\quad (\cong S^{m+n+1})$

by induction $S^k * S^0 \cong S^{k+1}$ as glueing of two $D^{k+1} \cong CS^k$ along their boundaries

$$\left\{(tx_1,\dots,tx_{k+1}) : \begin{array}{l} 0 \le t \le 1 \\ (x_1,\dots,x_{k+1}) \in S^k \end{array}\right\} \longleftrightarrow$$



quotient of $S^k \times I$

$$\| $$

$$\left\{(y_1,\dots,y_{k+1}) : \sum y_i^2 \le 1\right\} = D^{k+1} \subset \mathbb{R}^{k+1}$$