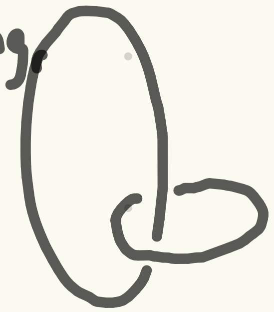
Chapter 1 Fundamental groups

Motivating example (<u>knot</u> <u>group</u> of the trivial knot)

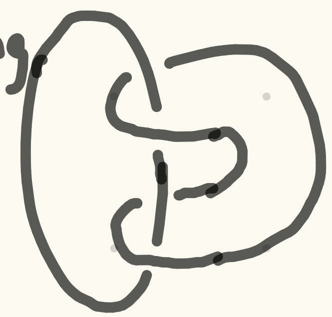
take a fixed "ring" in $\mathbb{R}^3$, and consider configurations of another loop around it

- identify configurations that are continuous deformations of each other (without passing through the ring)

ring



ring



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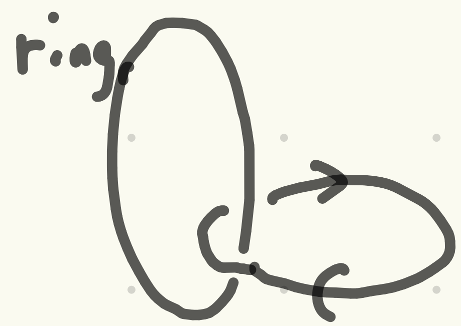
⇝ roughly speaking: the number of times the loop goes around the ring should be the (only) invariant

Put <u>orientation</u> on the loop and take it into account

vs.



once in counter-clockwise direction

once in clockwise dir.
= (−1)-th in counterclockwise dir.

product structure on oriented loops

AB : follow A first, then right before "closing" switch to B, and again right before "closing" B come back to the starting point in A

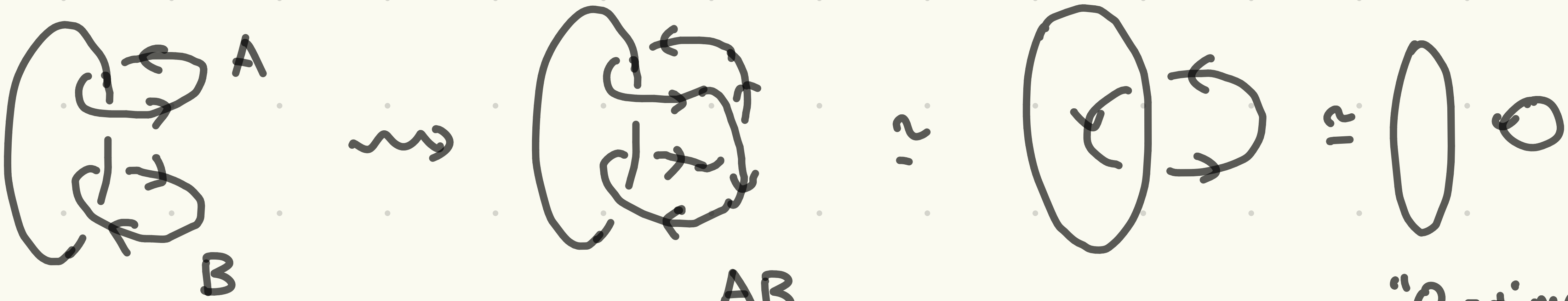
$\rightsquigarrow$ well-defined up to continuous deform.

cont.) if A goes around the ring a times

B ~ b times

then AB ~ $(a+b)$ times

all in counterclock direction



"0-times"
unit for product

$\Rightarrow$ get the group structure of $\mathbb{Z}$

Rem this group will be the fundamental group

$\pi_1(\mathbb{R}^3 \setminus \text{ring}, *)$

base point (does not matter)

Paths in topological spaces

X: topological space

$I = [0, 1]$

Def. a <u>path</u> in X is given by a continuous map $f : I \to X$

continuous deformation of paths

Def homotopy of paths :

$F : I \times I \longrightarrow X$ continuous map

parameter for cont. deform.

param. of individual paths

s.t. $F(0, t)$ is independent of t

$F(1, t)$ is also independent of t

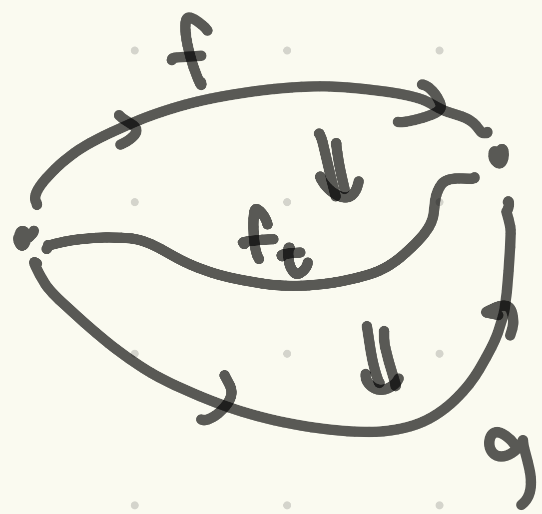
i.e. f, g paths are homotopic if

$\exists$ "continuous family" of paths f_t in X

$(f_t(s) = F(s, t))$ s.t.

(cont.)

- $f_0 = f$, $f_1 = g$
- f_t have the same starting point $f_t(0) = f(0)$ $(\Rightarrow g(0) = f(0))$
- $\sim$ same end point $f_t(1) = f(1)$



Caution: this is more restrictive than homotopy of maps (which has no condition about $F(0,t)$, $F(1,t)$)

Homotopy equivalence relation

Def. two paths f_0, f_1 are <u>homotopy equivalent</u> if $\exists$ homotopy of paths $F : I \times I \to X$ s.t.

$f_0(s) = F(s, 0)$, $f_1(s) = F(s, 1)$

write $f_0 \simeq f_1$ for this

Prop. homotopy equivalence relation is really an equivalence relation; i.e.,

- $f \simeq f$
- $f_0 \simeq f_1 \Leftrightarrow f_1 \simeq f_0$
- $f_0 \simeq f_1$, $f_1 \simeq f_2 \Rightarrow f_0 \simeq f_2$

Proof $f \simeq f$: $F(s,t) = f(s)$ does the job

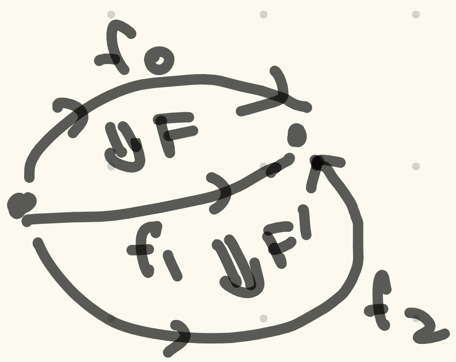
$f_0 \simeq f_1 \Rightarrow f_1 \simeq f_0$: suppose $F : I \times I \to X$

implements $f_0 \simeq f_1$ ($f_i(s) = F(s,i)$, $i = 0,1$)

$\rightsquigarrow$ "flip" the t-parameter $F'(s,t) = F(s,1-t)$

satisfies $F'(s,0) = f_1(s)$, $F'(s,1) = f_0(1)$

$f_0 \simeq f_1 , f_1 \simeq f_2 \Rightarrow f_0 \simeq f_2$:



take $F : I \times I \to X$ impl. $f_0 \simeq f_1$

$F' : I \times I \to X$ impl. $f_1 \simeq f_2$

do F at "double speed", then F'

at "double speed"

concretely $F''(s,t) = \begin{cases} F(s,2t) & (0 \le t \le \frac{1}{2}) \\ F'(s,2t-1) & (\frac{1}{2} \le t \le 1) \end{cases}$

well-defined as a map $I \times I \to X$

overlapping cases : $t = \frac{1}{2}$ $\rightsquigarrow$ compare

$F(s,1)$ and $F'(s,0)$, both $= f_1(s)$ OK

continuity : restriction of F'' to $I \times [0,\frac{1}{2}]$

and $I \times [\frac{1}{2}, 1]$ are continuous ; these closed

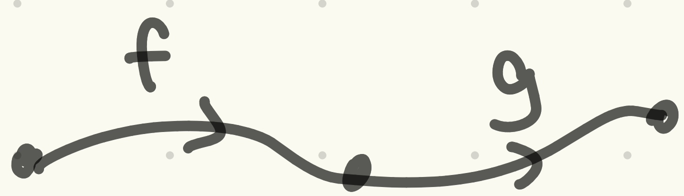
sets cover $I \times I$ $\Rightarrow$ OK

($F : Y \xrightarrow{\text{map}} X$, $A, B \subset Y$ closed, $F|_A, F|_B$ cont.

$\Rightarrow$ F is cont. on Y)

Terminology: the homotopy class of a path f in X: equiv. class for the homotopy equiv. relation; write $[f]$

Composition of paths



defined when $f(1) = g(0)$;

$f \cdot g$: follow f first, then g

concretely $f \cdot g : I \to X$,

$$(f \cdot g)(s) = \begin{cases} f(2s) & (0 \le s \le \frac{1}{2}) \\ g(2s-1) & (\frac{1}{2} \le s \le 1) \end{cases}$$

Prop. $[f\cdot g]$ only depends on $[f]$ and $[g]$

i.e. $f \simeq f',\ g \simeq g' \Rightarrow f\cdot g \simeq f'\cdot g'$

$$(f'(1) = f(1) = g(0) = g'(0))$$

Proof take $F, G : I \times I \to X$ implementing $f \simeq f'$ and $g \simeq g'$; combine them by

$$H(s,t) = \begin{cases} F(2s, t) & (0 \le s \le \frac{1}{2}) \\ G(2s-1, t) & (\frac{1}{2} \le s \le 1) \end{cases}$$

well-defined as a map: compare values at

$s = \frac{1}{2}$; $F(1,t)$ and $G(0,t)$

these are all $f(1) = g(0)$

continuity : $I \times I = \underbrace{([0,\tfrac{1}{2}] \times I)}\, \cup\, \underbrace{([\tfrac{1}{2},1] \times I)}$

closed, H is cont. on each

$H(s,0) = F(2s,0)$ or $G(2s-1,0) = (f \cdot g)(s)$

$H(s,1) = F(2s,1)$ or $G(2s-1,1) = (f' \cdot g')(s)$

so H implements $f \cdot g \simeq f' \cdot g'$ $\boxtimes$

Prop $f : I \to X$ path in X

define another path g by $g(s) = f(1-s)$

("flip the parameter") $g(0) = f(1)$, $g(1) = f(0)$

$[f \cdot g] = [\text{const. path at } f(0)]$

$[g \cdot f] = [\text{const. path at } f(1)]$

Proof for $[f \cdot g]$

not move from $f(0)$

Concretely: consider $H : I \times I \to X$ by

$$H(s,t) = \begin{cases} f(2ts) & (0 \le s \le \frac{1}{2}) \\ f(2t(1-s)) & (\frac{1}{2} \le s \le 1) \end{cases}$$

i.e. $f_t(s) = H(s,t)$ is the path

- go from $f(0)$ to $f(t)$ $(0 \le s \le \frac{1}{2})$
- then come back to $f(0)$ $(\frac{1}{2} \le s \le 1)$

$t = 0$ we get the const path at $f(0)$

$t = 1$ we get $f \cdot g$

$H(0,t) = f(0) = H(1,t)$ homotopy of path. $\boxtimes$

Rem $\quad$ (const. path at $f(0)) \cdot f \simeq f$

$$\underbrace{f \cdot (\text{const. path at } f(1))}_{\substack{\text{follow } f \text{ at twice the speed} \\ \text{then stay at } f(1)}} \simeq f$$

- So up to homotopy, the constant paths behave like units of composition
- the proposition says $g(s) = f(1-s)$ is the "inverse" of f

Def. $x_0 \in X$ "basepoint"

a <u>loop</u> around x_0 is a path $f: I \to X$ such that $f(0) = x_0 = f(1)$

The <u>fundamental</u> <u>group</u> $\pi_1(X, x_0)$ is

- as set : $\{f : \text{loop around } x_0\} /$ homotopy of paths
$= \{[f] : f : \text{loop around } x_0\}$
- group law : $[f][g] = [f \cdot g]$

so the const. path at x_0 is the unit.

Fundamental groupoid of X:

$\pi_1(X)$: groupoid with

- underlying "base" space X

"arrow" space $\{[f] : f : I \to X\}$
(path)

- product of arrows $[f][g] = [f \cdot g]$
defined only when $f(1) = g(0)$

i.e. $[f]$ has "source" $f(1)$, "range" $f(0)$

so $[f][g]$ defined when
(source of f) = (range of g)

$x \in X \rightsquigarrow$ const path e_x at x is the "unit at x"

$[e_x][f] = [f] = [f][e_y]$ if $f(0) = x$, $f(1) = y$

$[f \cdot g] = [e_x]$ for the "inverse" $g(s) = f(1-s)$

$[g \cdot f] = [e_y]$