Comparison of fundamental groups at diffent basepoints

X : topological space. $x_0, x_1 \in X$

caution : generally there is no relation between $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ because we can take $X = Y \amalg Z$, $x_0 \in Y$, $x_1 \in Z$ so

$\pi_1(X, x_0) = \pi_1(Y, x_0)$, $\pi_1(X, x_1) = \pi_1(Z, x_1)$

$\because$ loop starting from x_0 is connected

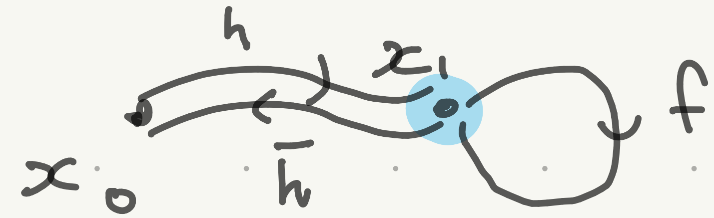
$\rightsquigarrow$ must be contained in $Y \subset X$

But : if x_0 and x_1 are in the same path-component

$\exists$ path $h : I \to X$, $h(i) = x_i$, $i = 0, 1$
$I = [0,1]$

then we set $\pi_1(X, x_0) \cong \pi_1(X, x_1)$

(cont.)



with $\bar{h}(s) = h(1-s)$ "inverse path" of h

$\pi_1(X, x_1) \to \pi_1(X, x_0)$, $[f] \mapsto [h f \bar{h}] = [h][f][\bar{h}]$

· group homomorphism because $[\bar{h}][h] = [\text{const path}]$ unit at x_1

· inverse homomorphism given by $[g] \mapsto [\bar{h} g h]$

Conceptually: $\pi_1(X, x)$: isotropy group G_x^x of the fundamental groupoid $G = \pi_1(X)$ $\{\gamma \in G : \text{"source } x_1\text{"}, \text{"range } x_0\text{"}\}$

$\quad = G_{x_1}^{x_0}$

generally G groupoid, $h \in G_{x_1}^{x_0}$ $\quad x_0, x_1 \in G^{(0)}$ base

$\rightsquigarrow \; G_{x_1}^{x_1} \xrightarrow{\sim} G_{x_0}^{x_0}$, $f \mapsto h f h^{-1}$ group isom.

$G_{x_1}^{x_1} = \{\gamma \in G : \text{source } \gamma = \text{range } \gamma = x_1\}$

Def. X is <u>simply</u> <u>connected</u> if $\forall x_0, x_1 \in X$

$\exists$ unique homotopy class path of paths from x_0 to x_1

↳ "at most one" (in another convention)

i.e., $\pi_1(X)$ has exactly one element from x_0 to x_1

equivalently

- X is path-connected (sometimes not assumed)

↳ then "at most" ..

- $\pi_1(X, x)$ is trivial for all $x \in X$

Rem $\pi_1(X, x_0) \curvearrowright \pi_1(X)_{x_1}^{x_0}$ $([f], [h]) \mapsto [f\,h]$

is a free and transitive action

$\Rightarrow$: multiply by $\bar{h}$

$f\,h \simeq h \iff f \simeq \bullet$

$x_0 \quad x_1$

const pat

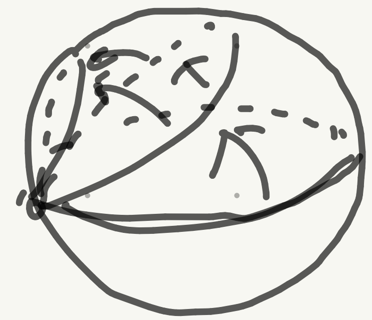
Ex. contractible $\Rightarrow$ simply connected

pt, $\mathbb{R}^n$, D^n, ...

$F : X \times I \to X$ homotopy between id_X and const. map

f : loop in X $\rightsquigarrow$ $F \circ f : f \simeq$ const.

S^2



S^n $n \geq 2$ in general

$SU(n)$ $(n \geq 2)$ $\{ X \in M_n(\mathbb{C}),\ X \overline{X^t} = I_n,\ \det X = 1 \}$

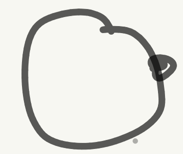
Next goal : $\pi_1(S^1, *) \cong \mathbb{Z}$

take the model $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\}$

$* = (1,0) \in S^1$

consider the loop $\omega : I \to S^1$, $s \mapsto (\cos 2\pi s, \sin 2\pi s)$

we want to prove



$*$

- any class in $\pi_1(S^1, *)$ is of the form $[\omega]^n$

  $= [\omega \cdot \dots \cdot \omega] = [(\cos 2\pi n s, \sin 2\pi n s)]$

  for some $n \in \mathbb{Z}$

- $[\omega]^n = e$ $(= [\text{const. path at } *]) \iff n = 0$

strategy : look at "lifts" as paths $I \xrightarrow{\tilde{f}} \mathbb{R}$

for loops $I \xrightarrow{f} S^1$ $\quad (f(s) = (\cos 2\pi \tilde{f}(s), \sin 2\pi \tilde{f}(s)))$

i.e. we look at $p : \mathbb{R} \to S^1$, $t \mapsto (\cos 2\pi t, \sin 2\pi t)$

and try to solve $f = p \circ \tilde{f}$

Def. X : topological space (base of covering)

a <u>covering space</u> of X is given by

- topological space $\tilde{X}$
- surjective continuous map $p : \tilde{X} \to X$

such that $\forall x \in X$ $\exists$ open set $U \ni x$:

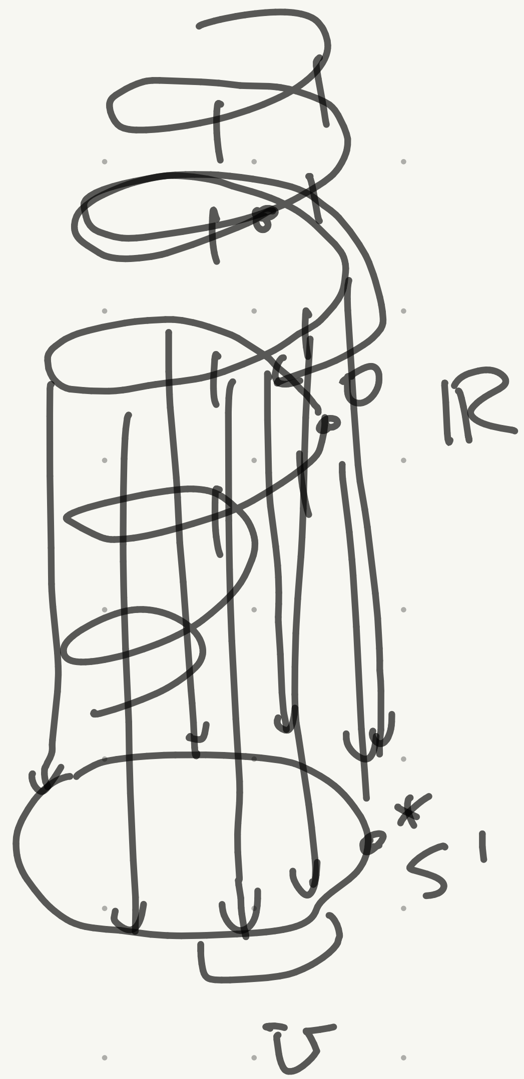
$$p^{-1}(U) = \bigcup_{i \in I} V_i \; ; \; V_i \subset \tilde{X} \text{ open}, \; V_i \cap V_j = \emptyset \quad (i \neq j)$$

$p|_{V_i} : V_i \to U$ is a homeomorphism for each i,

"locally" $\tilde{X}$ is $X \times$ (discrete set) ; $p^{-1}(U) \cong U \times I$

call such U an <u>evenly covered</u> open set.

Ex. above $p : \mathbb{R} \to S^1$ is a covering



$U \subset S^1$ open interval

$\{ (\cos 2\pi s, \sin 2\pi s) : a < s < b \}$

$|a - b| < 1$

$$p^{-1}(U) = \bigcup_{n \in \mathbb{Z}} \underbrace{(a+n, b+n)}_{V_n}$$

Prop $p : \tilde{X} \to X$ covering, $x_0 \in X$, $p(\tilde{x}_0) = x_0$. We have

a) if $f : I \to X$ is a path with $f(0) = x_0$ then

$\exists!$ $\tilde{f} : I \to \tilde{X}$ s.t. $\tilde{f}(0) = \tilde{x}_0$, $p \circ \tilde{f} = f$

unique

$\tilde{X}$

lift

other lift

b) if $F : I \times I \to X$ is a homotopy of paths

starting at x_0 $(F(0,t) = x_0)$ then

$\exists!$ $\tilde{F} : I \times I \to \tilde{X}$ homotopy of paths

starting at $\tilde{x}_0$ lifting F ; $F = p \circ \tilde{F}$

Prop. with $\tilde{S}^1 = \mathbb{R}$, $\tilde{x}_0 = 0 \Rightarrow \pi_1(S^1, *) \cong \mathbb{Z}$

Step 1 any $f : I \to S^1$ loop starting from $*$

is homotopic to $\omega_n : I \to S^1$ for some n

$s \mapsto (\cos 2\pi n s, \sin 2\pi n s)$

i.e. $[f] = [\omega]^n$

Prop a) gives a lift $\tilde{f} : I \to \mathbb{R}$ as a path

starting at $\tilde{x}_0 = 0 \in \mathbb{R}$ $(\tilde{f}(0) = 0)$

$\tilde{f}$ lifts $f \Rightarrow p(\tilde{f}(1)) = f(1) = * \Rightarrow \tilde{f}(1) \in \mathbb{Z}$

Put $n = \tilde{f}(1)$, and consider paths

$\tilde{\omega}_n : I \to \mathbb{R}$, $s \mapsto ns$ (lift of ω_n)

$\tilde{f}_t : I \to \mathbb{R}$, $s \mapsto (1-t)\tilde{f}(s) + t\tilde{\omega}_n(s)$

(cont.) then $(\hat{f}_t)_{t\in I}$ is a homotopy of paths

$\hat{f}_t(0) = (1-t)\,0 + t\,0 = 0$, $\tilde{f}_t(1) = (1-t)\underbrace{\tilde{f}(1)}_{n} + t\cdot\underbrace{\tilde{\omega}_n(1)}_{n} = n$

$f_t = p\circ\tilde{f}_t$ is a homotopy of paths between f and ω_n

Step 2 $[\omega_n]$ $(=[\omega]^n) = e = [\text{const. path}] \Rightarrow n = 0$

$[\omega_n] = [\text{const. path}]$ means there is a homotopy

$F\ \ I\times I \to S^1$ between ω_n and the const. path at $*$

Prop b) gives a (unique) lift $\hat{F} : I\times I \to \mathbb{R}$

starting at $\tilde{x}_0 = 0$; $\hat{f}_0(s) = \hat{F}(s,0)$ lifts ω_n,

$\tilde{f}_1(s) = \tilde{F}(s,1)$ lifts the const. path.

(cont.) by the uniqueness part of Prop. a),

$\tilde{f}_1 = \tilde{\omega}_n$, $\tilde{f}_0 = \tilde{\omega}_0$ = const. path at 0

then n $(= \tilde{\omega}_n(1) = \tilde{F}(1,0))$ should be $0 = \tilde{F}(1,1)$ $\boxtimes$

Proof of Prop. : a) and b) follow from :

c) $\forall$ Y, $F: Y \times I \to X$, $\tilde{f}_0 : Y \to \tilde{X}$ s.t. $p \circ \tilde{f}_0(y) = F(y,0)$

$\leftrightarrow$ "$(f_t : Y \to X)_{t \in I}$" "$f_0(y)$"

$\exists!$ $\tilde{F} : Y \times I \to \tilde{X}$ s.t. $p \circ \tilde{F} = F$, $\tilde{F}(y,0) = \tilde{f}_0(y)$

a): take $Y = pt$, $\tilde{f}_0 : pt \to \tilde{X}$, $* \mapsto \tilde{x}_0$ $(\mapsto f(0))$

b): take $Y = I$, $\tilde{f}_0 = \tilde{f}$ starting at $\tilde{x}_0$ (by a))

$\tilde{F}(0,t)$ is const ($p^{-1}(x_0)$ is Discrete) so is $\tilde{F}(1,t)$

in $\tilde{X}$, cont. in t $F(0,t) = x_0$

(cont.) take an evenly covered $U \ni x_0$

so $p^{-1}(U) = \bigcup_{i \in I} V_i$, $V_i \cap V_j = \emptyset$ $(i \neq j)$

$t \mapsto F(0,t)$ is a cont. map $I \to p^{-1}(U)$

so it must stay in $V_{i_0} \ni \tilde{x}_0 \Rightarrow F(0,t) = \tilde{x}_0$

similarly with $F(1,t)$ (stay const. at $\tilde{f}_0(1)$)

Proof of c)

Step 1 local construction of $\tilde{F}$ around $\underset{Y}{\{y_0\}} \times I$

First fix $t \in I$

take open neighborhoods $N_{y_0,t} \ni y_0$, $(a_{y_0,t}, b_{y_0,t}) \ni t$

s.t. $F(N_{y_0,t} \times \underbrace{(a_{y_0,t}, b_{y_0,t})}_{\text{to be precise } (I \cap \sim)}) \subset U$: evenly covered neigh. of $F(y_0,t)$

by compactness $\exists\, t_1, \dots, t_k$ s.t. $I \subset \bigcup_{i=1}^{k} (a_{t_i}, b_{t_i})$

then with $N_{y_0} = \bigcap_{i=1}^{k} N_{y_0, t_i}$, we have $0 = t'_0 \le \dots \le t'_m = 1$

s.t. $F(N_{y_0} \times [t'_i, t'_{i+1}]) \subset U_i$: evenly covered open set of X

i.e. $p^{-1}(U_i) = \bigcup_{j \in J} \tilde{U}^{(j)}_i$ $(\tilde{U}^{(j)}_i)_j$ disj. open sets

$p : \tilde{U}^{(j)}_i \to U_i$ homeo.

Construct $\tilde{F} : N_{y_0} \times I \to \tilde{X}$ as follows:

- on $N_{y_0} \times [t'_0, t'_1]$: choose (unique) j_0 s.t.

$\tilde{f}_0(y_0) \in \tilde{U}^{(j_0)}_0$, $\tilde{F} : N_{y_0} \times [t'_0, t'_1] \to \tilde{U}^{(j_0)}_0$

"$\tilde{y}_0$" lift of $F : N_{y_0} \times [t'_0, t'_1] \to U_0$

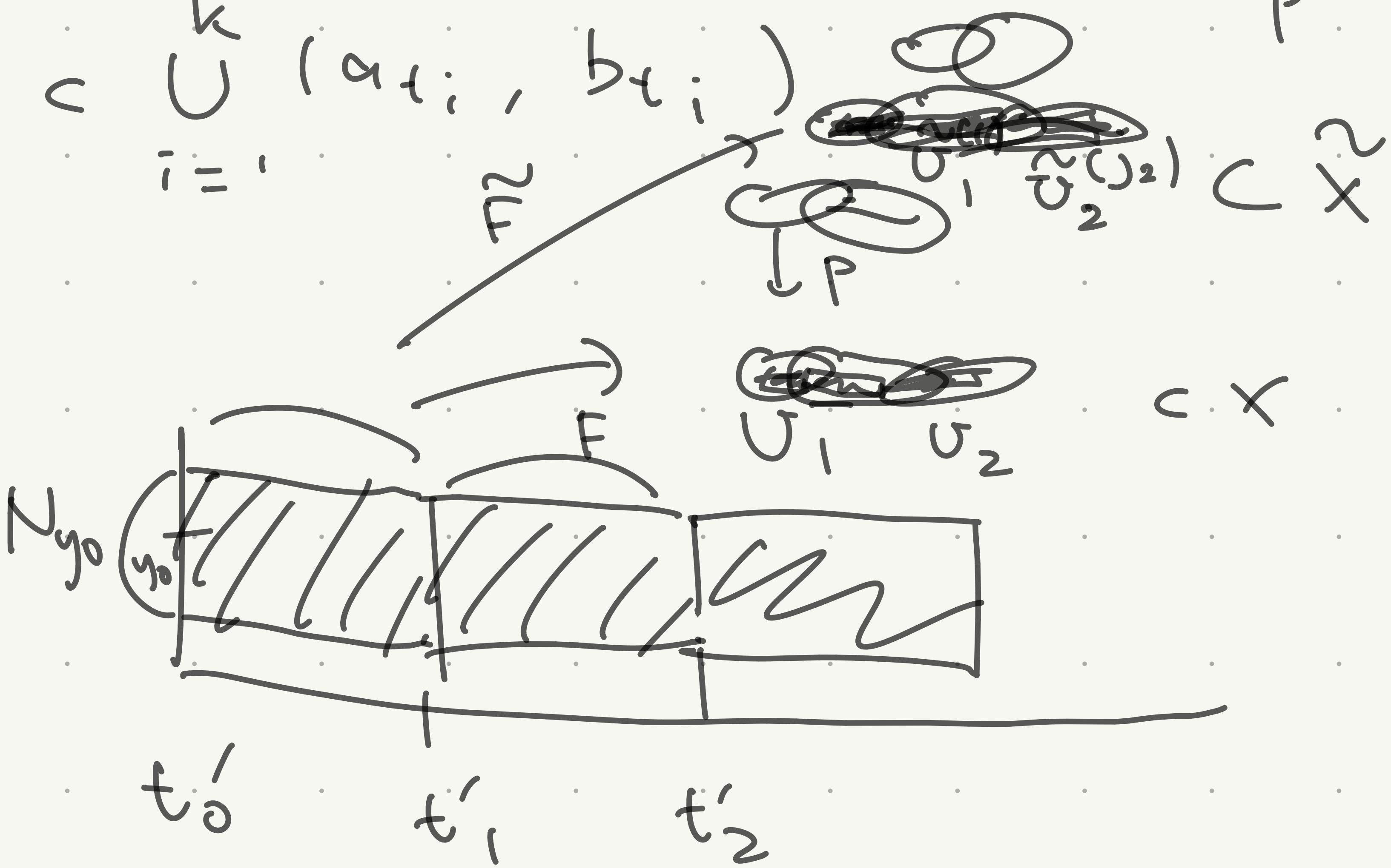
$$F(N_{y_0} \times (a_{t_i}, b_{t_i})) \subset U_i$$

evenly covered

for $i = 1, \dots, k$

$$p^{-1}(U_i) = \bigcup_{j \in I} V_j^{(i)}$$

$$I \subset \bigcup_{i=1}^{k} (a_{t_i}, b_{t_i})$$



- on $\{y_0\} \times [t_1', t_2']$: choose j_1 s.t. $\underbrace{\tilde{F}(y_0, t_1)}_{\text{we know from prev. step}} \in U_1^{(j_1)}$

define $\tilde{F} : N_{y_0} \times [t_1', t_2'] \to \tilde{U}_1^{(j_1)}$ as the lift of $F : N_{y_0} \times [t_1', t_2'] \to U_1$

$\rightsquigarrow$ continue up to $N_{y_0} \times [t_{m-1}', \underset{1}{\overset{\shortparallel}{t_m'}}]$. Then

- $\tilde{F} : N_{y_0} \times I \to \tilde{X}$ well-def'ed cont.
- $p \circ \tilde{F} = F$ on $N_{y_0} \times I$

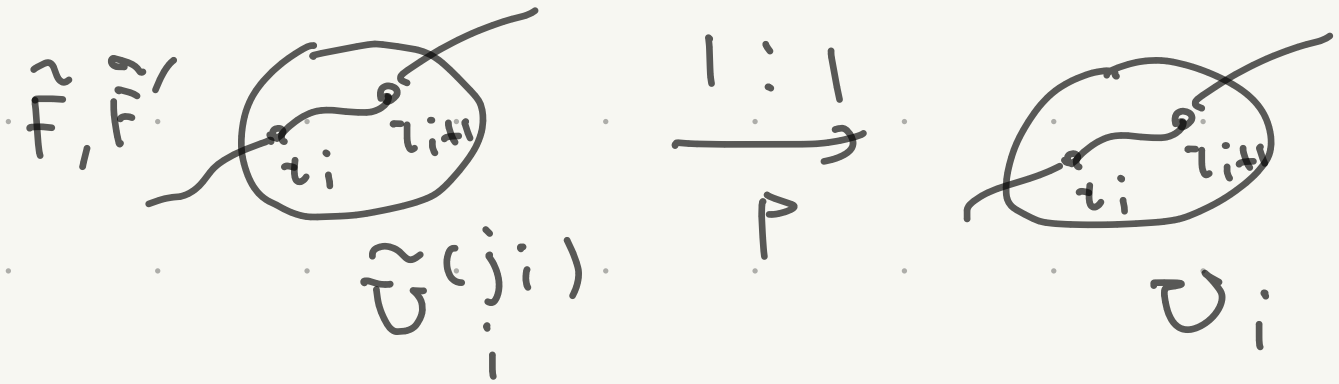
Step 2 uniqueness when $Y = \{y_0\}$

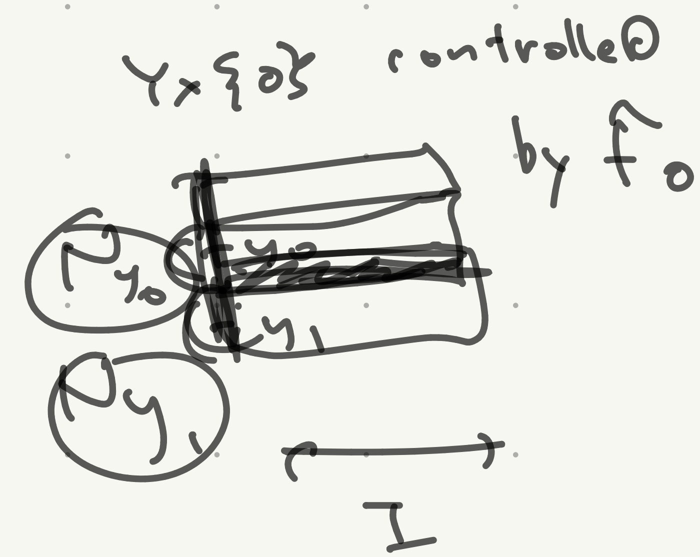
given another lift $\tilde{F}'$ starting at $\tilde{y}_0 = \tilde{f}_0(y_0)$,

choose $0 = t_0 \leq \cdots \leq t_m = 1$ as before by induction $\tilde{F} = \tilde{F}'$ on $\{y_0\} \times [0, t_i]$ because

both $\tilde{F}$ and $\tilde{F}'$ map $\{y_0\} \times [t_i, t_{i-1}]$ to $\tilde{U}^{(j_i)}_i$

(and they are lifts of $F|_{\{y_0\} \times [t_i, t_{i+1}]}$)





Step 3 $\tilde{F}$ well-defined for general Y

we know $\left(\begin{array}{l}\text{how to construct } \tilde{F}|_{N_{y_0} \times I} \quad \forall\, y_0 \\ \text{restriction to } \{y_0\} \times I \text{ is unique}\end{array}\right.$

$\Rightarrow$ if $N_{y_0} \cap N_{y_1} \neq \emptyset$, two defs of $\tilde{F}$ on $(N_{y_0} \cap N_{y_1}) \times I$ agree ☐

Remarks • Prop c) can be expressed as

$$\begin{tikzcd} y \arrow[d, mapsto] & Y \arrow[r, "\tilde{f}_0"] \arrow[d] & \tilde{X} \arrow[d, "p"] \\ (y,0) & Y \times I \arrow[r, "F"'] \arrow[ur, dashed, "\exists!\, \tilde{F}"] & X \end{tikzcd}$$

p : covering (commutative)

in other words, covering maps have the homotopy lifting property with respect to any space Y;

or covering maps are fibrations

• $S^1 \cong \mathbb{R}/\mathbb{Z}$ for the proper action $\mathbb{Z} \curvearrowright \mathbb{R}$

contractible

this is a particular case of the classifying space

Interpretation of $\pi_1(S^1, *) \cong \mathbb{Z}$:

the maps $S^1 \to S^1$, $* \mapsto *$ are classified (up to basepoint-fixing cont. deform.) by their <u>winding numbers</u> (how many times does a loop "go around")

Application : Brouwer's fixed point theorem

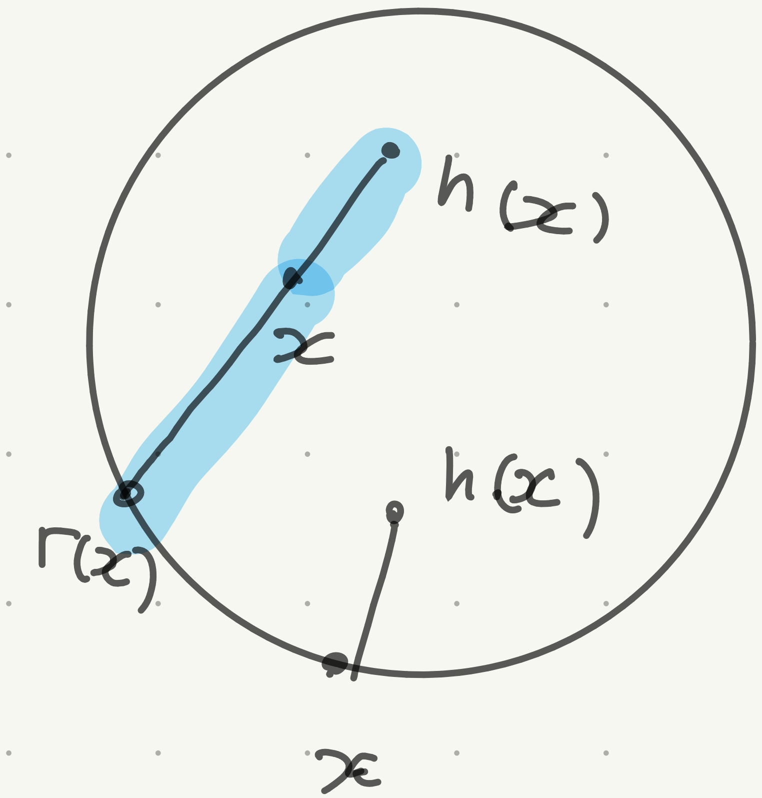
any continuous map $h : D^2 \to D^2$ has a fixed point
closed 2-disk

i.e. $\forall h \ \exists x_h \in D^2 : h(x_h) = x_h$

Proof by contradiction : if there is no such x_h, we would have $r : D^2 \to S^1$, $r(x) = x$ for S^1
(retraction of $S^1 \hookrightarrow D^2$)

- construction of r : $r(x) \in S^1$ is the unique point s.t. x is on the line segment between $h(x)$ and $r(x)$ (well-defined by $x \neq h(x)$)

  note : $x \in S^1 \Rightarrow r(x) = x$

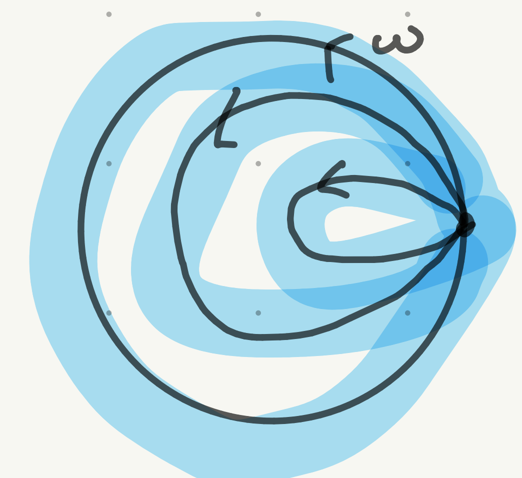


- why this leads to a contradiction :

  $\omega : I \to S^1$ $(s \mapsto (\cos 2\pi s, \sin 2\pi s))$ would be homotopic to the const. loop ;

  in D^2 ω is contractible by $F : I \times I \to D^2$

  $F(s,t) = (1-t)\,\omega(s) + t(1,0)$



  then $r \circ F$ gives $\omega \simeq$ const.
  $I \times I \to S^1$ $\Rightarrow [\omega] = e$ in $\pi_1(S^1, *)$