Another application of $\underset{S^1}{\pi_1(\mathbb{R}/\mathbb{Z}, *)} \cong \mathbb{Z}$

Borsuk-Ulam th'm) $f_1, f_2 : S^2 \to \mathbb{R}$ cont. funcs
(Th'm. 1.10)
f_1: "temperature", f_2: "humidity"

then $\exists\, x \in S^2$ s.t. $f_1(x) = f_1(\underbrace{-x}_{\text{as pt. in } \mathbb{R}^3})$ and $f_2(x) = f_2(-x)$

Proof by contradiction: $f = (f_1, f_2) : S^2 \to \mathbb{R}^2$

suppose $\nexists\, x \in S^2$ $f(x) = f(-x)$

then $g(x) = \dfrac{f(x) - f(-x)}{|f(x) - f(-x)|}$ is well-defined as

cont. map $S^2 \to S^1 \subset \mathbb{R}^2$; note $g(-x) = -g(x)$

consider the loop $\eta : I \to S^2$, $s \mapsto (\cos 2\pi s, \sin 2\pi s, 0)$

"go around the equator"

$h = g\eta : I \to S^1$ induced loop in S^1

take its lift $\tilde{h} : I \to \mathbb{R}$ s.t. $\tilde{h}(0) = 0$

(uniquely exists by Prop. from last time)

$\eta(s + \frac{1}{2}) = -\eta(s)$ $(0 \le s \le 1)$ and $g(-s) = -g(s)$

implies $\tilde{h}(s + \frac{1}{2}) = \tilde{h}(s) + \frac{q}{2}$ for some $q \in 2\mathbb{Z}+1$

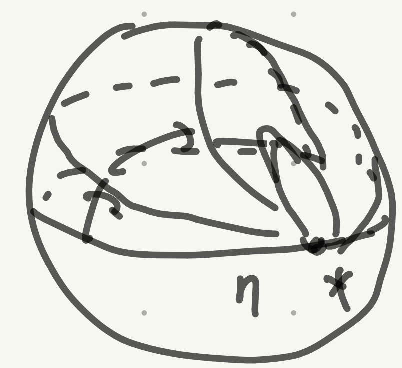
$(p : \mathbb{R} \to S^1,\ t \mapsto (\cos 2\pi t, \sin 2\pi t);\ h = p\tilde{h})$

$\underbrace{\tilde{h}(1)}$ $= \tilde{h}(\frac{1}{2}) + \frac{q}{2} = \tilde{h}(0) + q = q$: odd. int.

int. representing

$[h] \in \pi_1(S^1, *) \cong \mathbb{Z}$

but η is contractible in S^2



$\Rightarrow$ so is $g\eta$ $\Rightarrow$ $q = 0$ contradicts

with $q \in 2\mathbb{Z}+1$

Fundamental group of direct product

Prop (1.12) $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$

Proof (sketch) $f(z) = (g(z), h(z))$

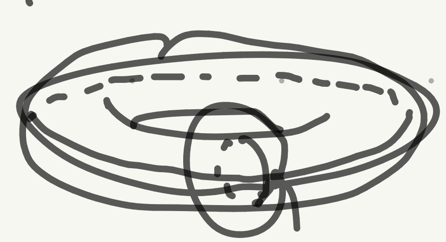
generally: continuous maps $Z \xrightarrow{f} X \times Y$

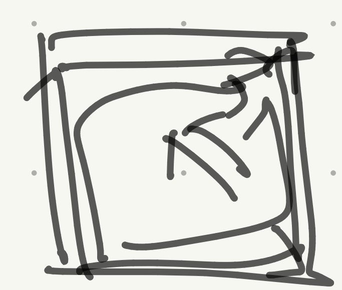
$\equiv$ pairs of cont. maps $Z \xrightarrow{g} X$, $Z \xrightarrow{h} Y$

- paths in $X \times Y$ $\equiv$ paths in X & paths in Y
  $(Z = I)$

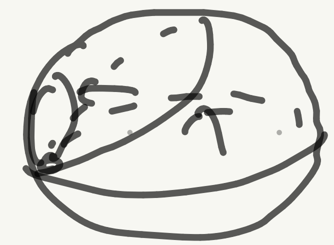
- homotopy of paths in $X \times Y$
  $(Z = I \times I)$ $\equiv$ componentwise homotopy

Example $\pi_1(S^1 \times S^1, x) \cong \mathbb{Z}^2$

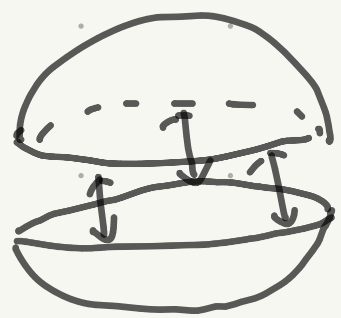




Prop 1.14 $\pi_1(S^n, *)$ is trivial for $n \geq 2$

higher dim version of "  "

strategy: $S^n \cong (D^n \amalg D^n) /$ identify corresponding points on $S^{n-1} = \partial D^n$



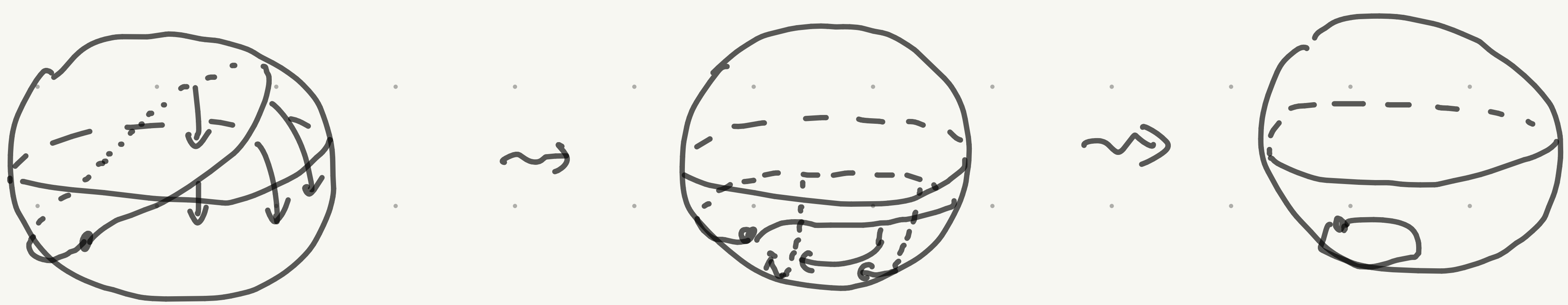
D^2

bdry S^1 $\rightsquigarrow$ S^2

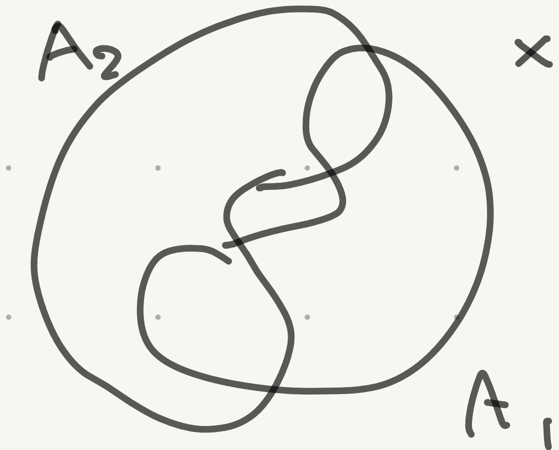
D^2

any loop in S^n can be "compressed" to a loop in one copy of D^n

$\rightsquigarrow$ use contraction of loops in D^n



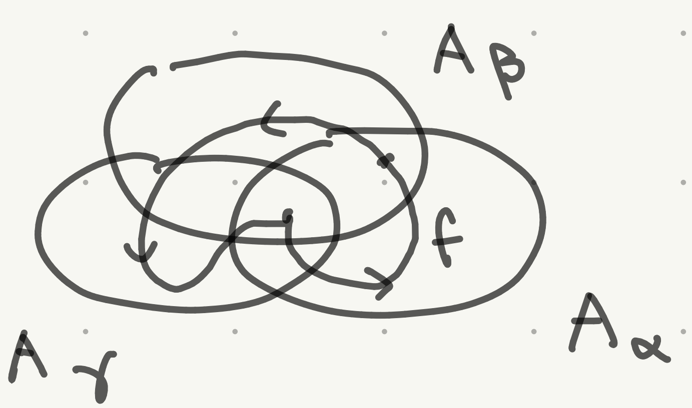
More formally:

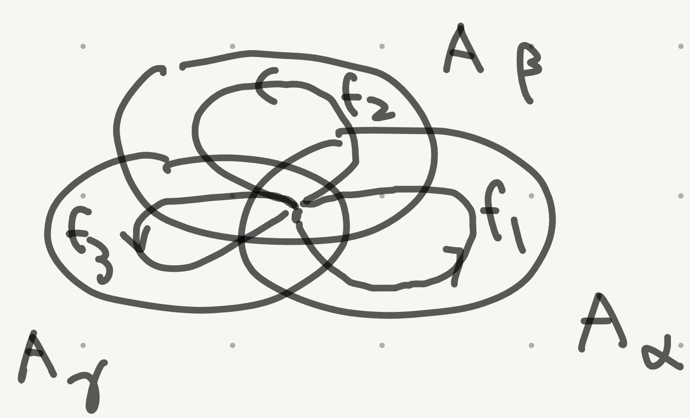


Lem. 1.15 If $X = \bigcup_{\alpha \in I} A_\alpha$ with path-connected open sets A_α s.t.

- $A_\alpha \cap A_\beta$ also path-conn. for $\alpha, \beta \in I$,
- $x_0 \in \bigcap_{\alpha \in I} A_\alpha$,

then any loop f in X starting from x_0 is homotopic to some $f_1 \cdot \dots \cdot f_k$, f_i loop in some A_{α_i} starting from x_0



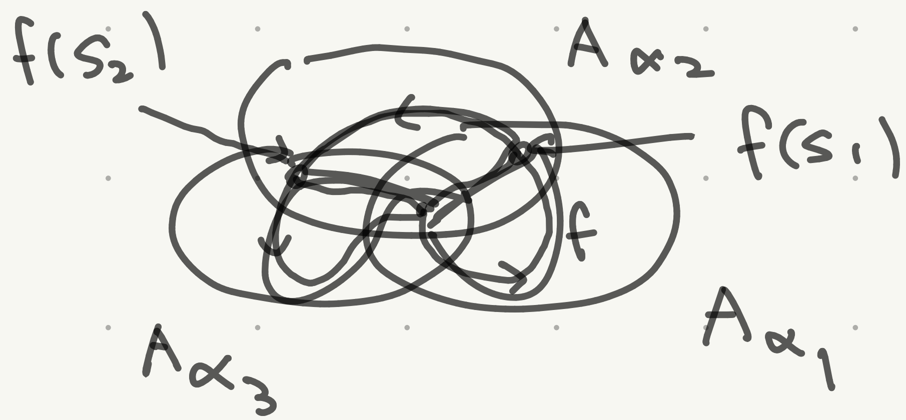


Proof choose $0 \leq s_0 \leq \cdots \leq s_k = 1$ s.t.

$f([s_{i-1}, s_i])$ is contained in some A_{α_i}

(possible by compactness of I)

$g_i : I \to A_{\alpha_i} \cap A_{\alpha_i}$ path connecting x_0 to $f(s_i)$



put $f_i = g_{i-1} \cdot (f|_{[s_{i-1}, s_i]}) \cdot \overline{g_i}$

flipped path

this is a loop in A_{α_i}

$$f \simeq \underbrace{(f|_{[0,s_1]} \cdot \overline{g_1})}_{f_1} \cdot \underbrace{(g_1 \cdot f|_{[s_1,s_2]} \cdot \overline{g_2})}_{f_2} \cdot \cdots \cdot \underbrace{(g_{k-1} \cdot f|_{[s_{k-1},1]})}_{f_k}$$

$\boxtimes$

Proof of Prop 1.14

$S^n = A_1 \cup A_2$, A_i: "small open neighborhood" of D^n at "hemisphere"

e.g. $A_1 = \{ (x_1, \cdots, x_{n+1}) \in S^n \subset \mathbb{R}^{n+1}, \ x_{n+1} > -0.1 \}$

$A_2 = \{ (x_1, \cdots, x_{n+1}) \in S^n \subset \mathbb{R}^{n+1}, \ x_{n+1} < 0.1 \}$

So $A_1 \cap A_2$ is a "small neighborhood" of $S^{n-1} \subset S^n$ at "equator"

Lem $\Rightarrow$ any class in $\pi_1(S^n, \overset{\shortparallel}{*})$ is of the form
$(1, 0, \cdots, 0)$

$[f_1] \cdots [f_k]$, f_i loop in A_1 or A_2

each f_i is homotopic to const. loop in A_i

⋮

Cor $S^n \not\cong S^1$ for $n \neq 1$

(in fact, $S^m \cong S^n \iff m = n$)

Q. how do we improve this to $S^n \not\simeq S^1$?

not homotopic

$\rightsquigarrow$ we need to sort out the relation between homotopy and fundamental group

Functoriality of π_1

observation : fundamental group is defined for (X, x_0) pointed space (top. sp, its pt.)

map of pointed spaces : $(X, x_0) \xrightarrow{\varphi} (Y, y_0)$

$\equiv$ cont. map $\varphi : X \to Y$ s.t. $\varphi(x_0) = y_0$

$\rightsquigarrow$ map $\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by $[f] \mapsto [\varphi f]$

- well-defined : $f_0 \simeq f_1 \Rightarrow \varphi \circ f_0 \simeq \varphi \circ f_1$

- group-hom : $\varphi \circ (f_0 \cdot f_1) = \underbrace{(\varphi \circ f_0) \cdot (\varphi \circ f_1)}$

compos. as cont. map

$\underbrace{s \mapsto \varphi(f_0 \cdot f_1(s))}$

"compos." as path $f_0 \frown f_1$

this assignment $\varphi \rightsquigarrow \varphi_*$ is <u>functorial</u> :

- $(\varphi \circ \psi)_* = \varphi_* \circ \psi_*$ for $(X, x_0) \xrightarrow{\psi} (Y, y_0) \xrightarrow{\varphi} (Z, z_0)$
- $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X, x_0)}$

i.e. $(X, x_0) \mapsto \pi_1(X, x_0)$, $\varphi \mapsto \varphi_*$ is a <u>functor</u>
(pointed spaces) $\to$ (groups)

Rem without specifying basepoints,

$\varphi : X \to Y$ induces groupoid hom. $\pi_1(X) \xrightarrow{\varphi_*} \pi_1(Y)$

Prop 1.17 If $A \subset X$ is a retract i.e., $\exists\, r : X \to A$
s.t. $r\, i = \mathrm{id}_A$ then $i_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ $(x_0 \in A)$ is injective

i incl. map $A \to X$

If $A \subset X$ is a deformation retract $(i r \simeq \mathrm{id}_X)$
then i_* is an isomorphism

Homotopy for pointed spaces

we want to make sense of cont. deformation of maps $\varphi_t : (X, x_0) \to (Y, y_0)$ so that $(\varphi_0)_* = (\varphi_1)_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$, etc.

Def. a basepoint-preserving homotopy of maps between pointed spaces $(X, x_0), (Y, y_0)$: given by

$\Phi : X \times I \to Y$ cont. s.t. $\Phi(x_0, t) = y_0$

(more generally: for pairs $(X, A), (Y, B)$
$\Phi : X \times I \to Y$, $\Phi(A \times I) \subset B$)

$(X, x_0) \simeq (Y, y_0)$ if $\exists$ $(X, x_0) \underset{g}{\overset{f}{\rightleftarrows}} (Y, y_0)$
homotop. inv. to each other by above

so $\varphi_0 \simeq \varphi_1$ by basepoint-pres. homotopy

$\Rightarrow (\varphi_0)_* = (\varphi_1)_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$

we can relax this a bit:

Prop. 1.18 $\varphi : X \to Y$ homotopy equivalence

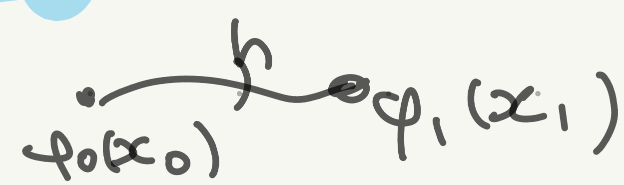
($so\ \exists\ \psi : Y \to X,\quad \varphi\psi \simeq id_Y,\ \psi\varphi \simeq id_X$)

then $\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, \varphi(x_0))$ is an isomorph.

Proof Step 1 suppose $(\varphi_t)_{t \in I}$ is a cont. family

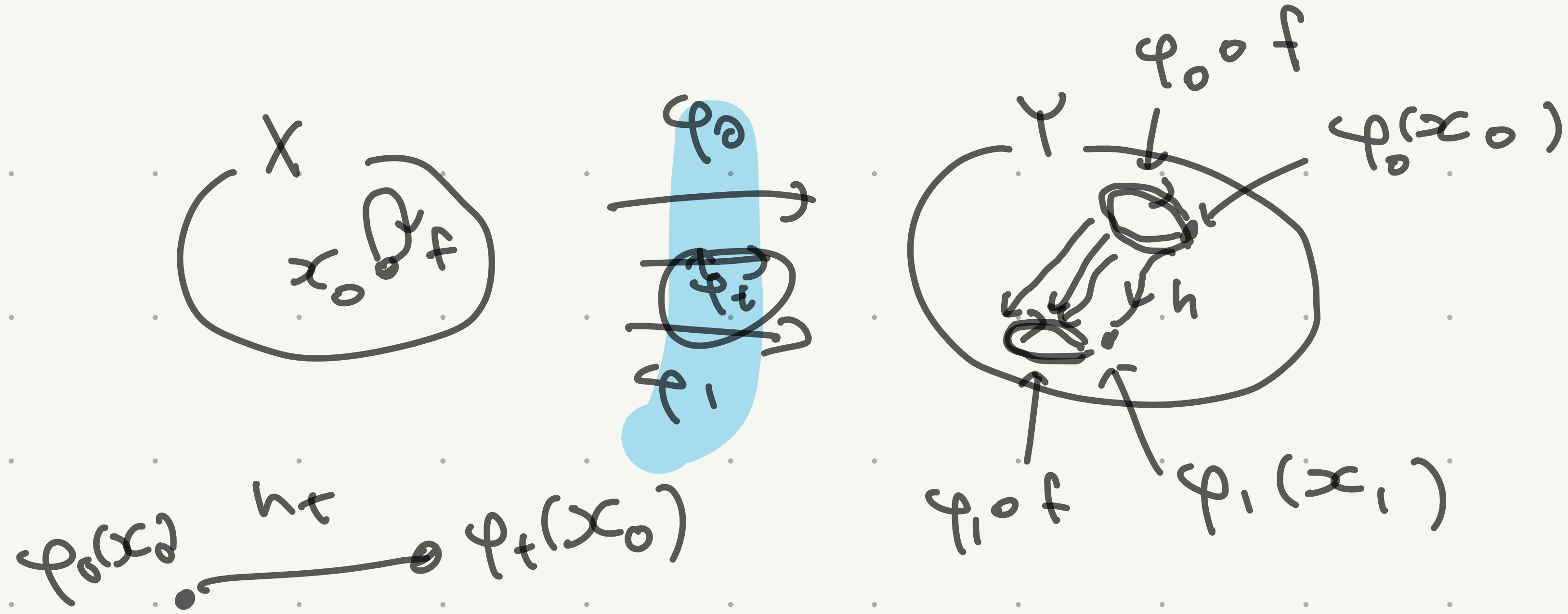
of maps $X \to Y$ (from $\Phi : X \times I \to Y$)

$h(s) = \varphi_s(x_0) = \Phi(x_0, s)$ path in Y



then $(\varphi_0)_*([f]) = \underbrace{[h](\varphi_1)_*([f])[\bar{h}]}$ for $[f] \in \pi_1(X, x_0)$

(cont.)



with $h_t(s) = h(ts)$ $\quad h_t \cdot (\varphi_t \circ f) \cdot \overline{h_t}$ interpolates $\varphi_0 \circ f$ and $h \cdot (\varphi_1 \circ f) \cdot \bar{h}$ (as homotopy of loops starting at $\varphi_0(x_0)$)

$$\Rightarrow (\varphi_0)_*([f]) = [\varphi_0 \circ f] = [h \cdot \varphi_1 \circ f \cdot \bar{h}] = [h]\,(\varphi_1)_*([f])\,[\bar{h}]$$

Step 2 $\quad \psi_* \varphi_* : \pi_1(X, x_0) \to \pi_1(X, \psi\varphi(x_0))$ is isom.

by Step 1 $\quad \underline{(\psi\varphi)_*([f])} = [h][f][\bar{h}] = [h][f][h]^{-1}$

for h connecting x_0 to $\psi\varphi(x_0)$

$\psi\varphi \simeq id_X$

we got $\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, \varphi(x_0))$ inj.

$\psi_* : \pi_1(Y, \varphi(x_0)) \to \pi_1(X, \psi\varphi(x_0))$ surj.

$$\downarrow 2$$

$$\pi_1(X, x_0)$$

Step 3 $\psi_* = \varphi_*^{-1}$ up to $\pi_1(X, \psi\varphi(x_0)) \simeq \pi_1(X, x_0)$

coming from path connecting x_0 and $\psi\varphi(x_0)$

Switch the roles of φ and ψ in Step 2 $\boxtimes$

fundamental groups of topological groups

topological group G

- top. sp.
- group

$G \times G \xrightarrow{m} G$, $(g, h) \mapsto gh$

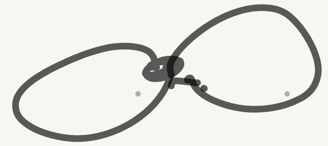
$G \to G$, $g \mapsto g^{-1}$

continuous

Ex. • $S^1 \subset \mathbb{C}$ group by product of complex numbers

- matrix groups $GL_n(\mathbb{R})$, $U(n)$, ...
- $\mathbb{R}^n$, ...

Th'm $\pi_1(G, e)$ is commutative

(Cor  has noncomm. fund. group

$\rightsquigarrow$ cannot be top. group)

Ex. $\pi_1(S^1, *) \cong \mathbb{Z}$ comm. ...

Key observation

$$\pi_1(G,e) \times \pi_1(G,e) \cong \pi_1(G \times G, (e,e)) \xrightarrow{m_*} \pi_1(G,e)$$

is equal to the product map of $\pi_1(G,e)$

(#)

i.e. prod. map on $\pi_1(G,e)$ is group hom

$\Rightarrow$ $\pi_1(G,e)$ is commutative

(*)

$(*)$: $m_{\pi_1(G,e)} : \overbrace{\pi_1(G,e) \times \pi_1(G,e)}^{\text{direct prod. group}} \to \pi_1(G,e)$

$$([f_1], [f_2]) \mapsto [f_1 \cdot f_2]$$

$$\shortparallel$$

$$([f_1], e_{\pi_1(G,e)}) \cdot (e_{\pi_1(G,e)}, [f_2])$$

$$\shortparallel$$

$$(e_{\pi_1(G,e)}, [f_2]) \cdot ([f_1], e_{\pi_1(G,e)})$$

$$\downarrow m \qquad\qquad \downarrow m$$

$$[f_2] \cdot [f_1] = [f_2 \cdot f_1]$$

$(\#)$: $G \underset{i_1}{\overset{m_G}{\rightleftarrows}} G \times G$, $g \mapsto (g, e)$ m gives retract

$\leadsto$ $(i_1)_*: \pi_1(G,e) \to \pi_1(G \times G, (e,e))$ is a group

(cont.)

hom s.t. $(m_G)_*((i_1)_*([f])) = [f]$

similar with $i_2 : G \to G \times G$, $g \mapsto (e, g)$

$([f_1], [f_2]) = (i_1)_*([f_1]) \cdot (i_2)_*([f_2])$

prod. in $\pi_1(G, e) \times \pi_1(G, e)$

under $(m_G)_*$, the right hand side goes to

$(m \circ i_1)_*([f_1]) \cdot (m \circ i_2)_*([f_2]) = [f_1][f_2]$