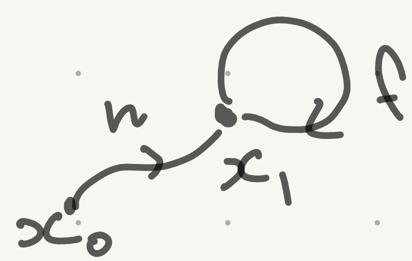
Exercise set 2 Section 1.1

Problem 3

X : path-connected

$\pi_1(X, x_0)$ is abelian for $\forall / \exists\ x_0 \in X$

$\Leftrightarrow$ $\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$, $[f] \mapsto [h \cdot f \cdot \bar{h}]$



is independent of the choice of h (among the paths from x_0 to x_1) for all $x_0, x_1 \in X$

Step 1 G : group

G is abelian $\Leftrightarrow$ $\forall g \in G$ $\alpha_g = id_G$ for the map $\alpha_g \in Aut(G)$, $\alpha_g(h) = ghg^{-1}$

$\because$ $gh = hg \Leftrightarrow ghg^{-1} = h$

Step 2 $\quad h, h'$: paths from x_0 to x_1

$$\beta_h = \beta_{h'} \iff \alpha_{[h\cdot \bar{h'}]} = id_{\pi_1(X, x_0)}$$

$$\because \alpha_{[h\cdot\bar{h'}]}([f]) = [\underbrace{h\cdot\underbrace{\bar{h'}\cdot f\cdot h'}_{\beta_{h'}^{-1}}\cdot \bar{h}}_{\beta_h}] = \beta_h \beta_{h'}^{-1}([f])$$

for $[f] \in \pi_1(X, x_0)$ represents $[h\cdot\bar{h'}]^{-1}$

Step 3 $\quad$ any $[f] \in \pi_1(X, x_0)$ is of the form $[h\cdot\bar{h'}]$

for some paths h, h' as above

$\because$ by the path-connectedness of X we can choose

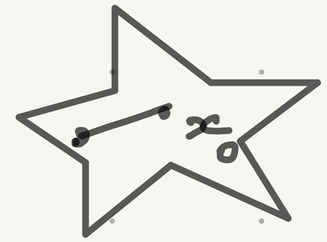
a path h' from x_0 to x_1

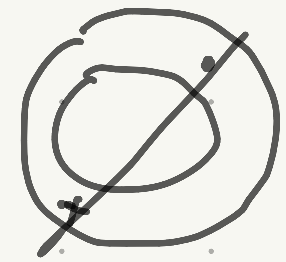
$$\Rightarrow [f] = [\underbrace{f\cdot h'}_{\text{take this as } h}\cdot \bar{h'}]$$

cancelled

Problem 4

star-shaped :





not star-shaped

or

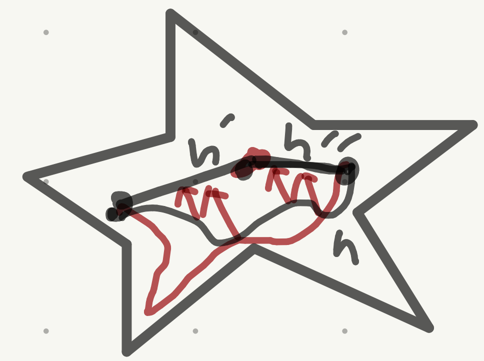
Step 1 $\quad X \subset \mathbb{R}^n$ star-shaped with basepoint x_0

h : path in X, from x_1 to x_2

$\Rightarrow$ h is homotopic to the piecewise linear path $h' \cdot h''$

h' : linear path from x_1 to x_0

h'' : linear path from x_0 to x_2

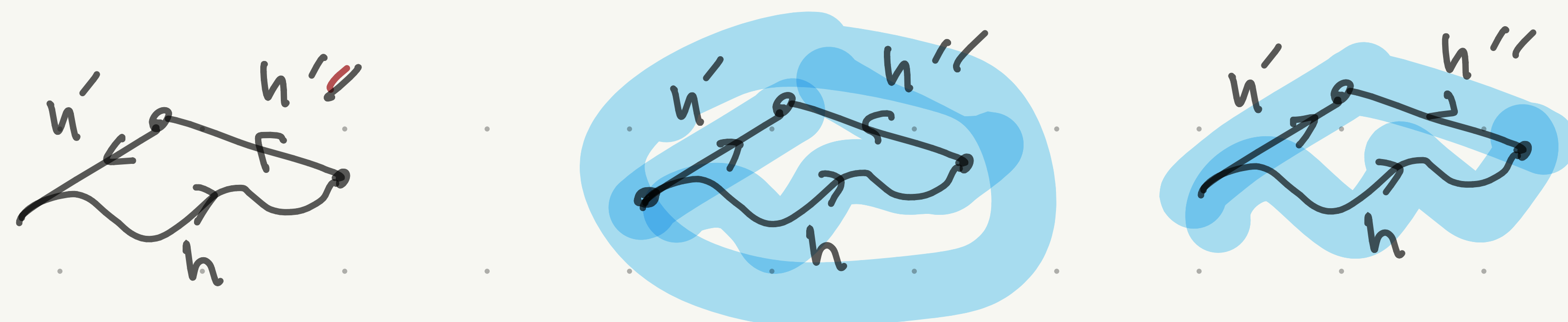


$\because$ Claim : $f = \overline{h'} \cdot h \cdot \overline{h''}$ is homotopic to the const. path at x_0

$$f_t(s) = (1-t)\,f(s) + t\,x_0$$ (convex combination in $\mathbb{R}^n$) stays in X

(cont.) $\overline{h'} \cdot h \cdot \overline{h''} \simeq$ const. path (claim)

$\Rightarrow h \cdot \overline{h''} \simeq h' \Rightarrow h \simeq h' \cdot h''$



Step 2 $X \subset \mathbb{R}^n$ locally star-shaped

$\Rightarrow$ any path f in X is homotopic to a piecewise linear path $h_1 \cdot \cdots \cdot h_m$ h_i: linear path from x_i to x_{i+1}

$\because$ $f : I \to X$ cont. map

By the compactness of I $\exists\ 0 = s_1 \le s_2 \le \cdots \le s_{n+1} = 1$ s.t. $f([s_i, s_{i+1}])$ is contained in some star-shaped subset of X

(cont.) Put $x_i = f(s_i) \in X$

By Step 1, $f|_{[s_i, s_{i+1}]}$ (path from x_i to x_{i+1})

is homotopic to a piecewise lin. path h_i'

$\Rightarrow f \simeq \underbrace{h_1' \cdot h_2' \cdot \cdots \cdot h_n'}_{\text{piecewise linear}}$

Step 3 $X \subset \mathbb{R}^n$ open $\Rightarrow$ locally star-shaped

$\because$ ε-balls in $\mathbb{R}^n$ are star-shaped

Step 4 $X = A_1 \cup \cdots \cup A_k$, $A_k \subset \mathbb{R}^n$ closed convex

$\Rightarrow$ locally star-shaped

fix $x_0 \in X$, put $\delta_i = \operatorname{dist}(x_0, A_i) = \inf\limits_{x \in A_i} d(x_0, x)$

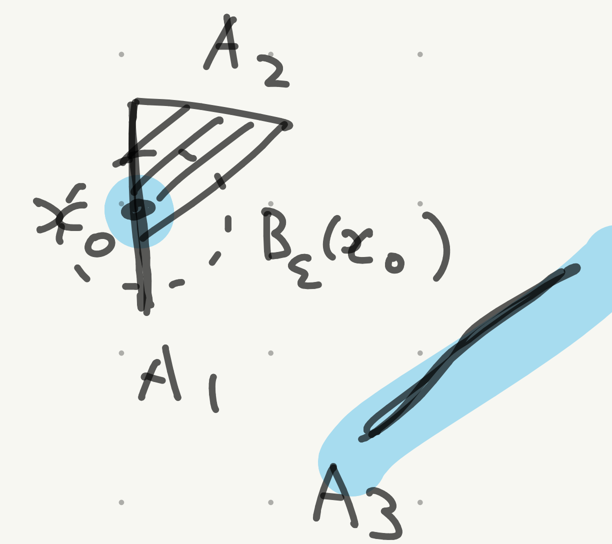
A_i closed $\Rightarrow$ $\delta_i > 0$ unless $x_0 \in A_i$

choose $\varepsilon > 0$ s.t. $\varepsilon < \min(\delta_i : \delta_i > 0)$

so $B_\varepsilon(x_0) \cap A_i = \emptyset$ if $x_0 \notin A_i$

ε-ball around x_0

$$U = \bigcup_{i=1}^{k} \left(B_\varepsilon(x_0) \cap A_i \right) = B_\varepsilon(x_0) \cap X$$



is a neighborhood of x_0 in X

generic pt. in U $\to$ $y \in B_\varepsilon(x_0) \cap A_i$ $\Rightarrow$ the line segment between x_0 and y belongs to $B_\varepsilon(x_0) \cap A_i$

$\Rightarrow$ U is star-shaped

"conterexample" when there are infinitely many components $X = \bigcup_{i=0}^{\infty} A_i$

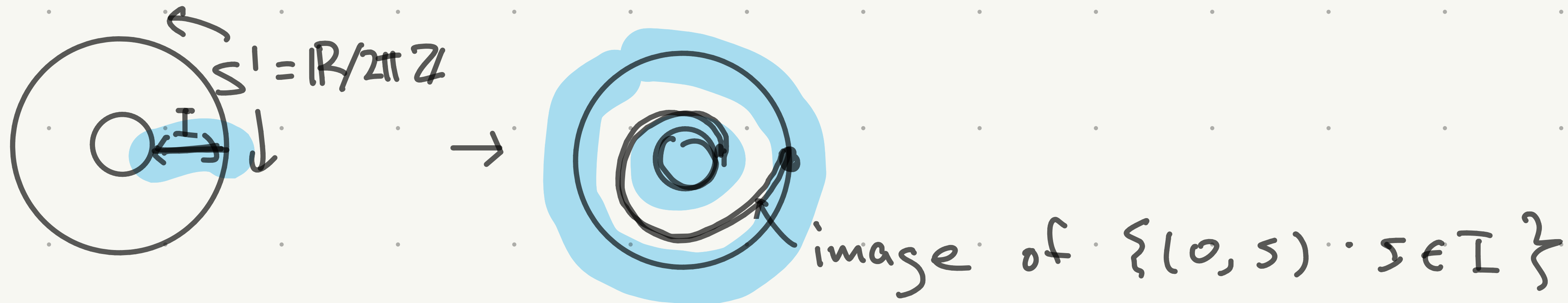
$A_0 = \{(x,0) : x \in \mathbb{R}\} \subset \mathbb{R}^2$: "x-axis"

$A_n = \{(x, \frac{1}{n}) : x \in \mathbb{R}\}$ $n = 1, 2, \cdots$

any neighborhood of $(x,0)$ contains $(x', \frac{1}{n})$ for $n \gg 1$

$\rightsquigarrow$ convex comb. goes outside of X

Problem 7

$f:$ 

homotopy $f \simeq id_{S^1 \times I}$ which is identity on the "inner boundary" $S^1 \times \{0\}$

$F : \underbrace{S^1 \times I}_{\text{input for } f,\, f_t,\, \dots} \times I \to S^1 \times I, \quad (\theta, s, t) \mapsto (\theta + 2\pi t s, s)$

I ← param. t for deform.

well-defined, cont.,
$F(\theta, s, 0) = (\theta, s) \quad (id_{S^1 \times I})$

$F(\theta, s, 1) = f(\theta, s)$

always identity on $S^1 \times \{0\}$: $F(\theta, 0, t) = (\theta, 0)$

homotopy $f \simeq id_{S^1\times I}$ which is identity on the "outer boundary" $S^1\times\{1\}$

$F : S^1\times I\times I \longrightarrow S^1\times I$, $(\theta, s, t) \mapsto (\theta + 2\pi(1-t(1-s)), t)$

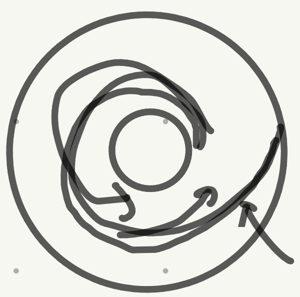
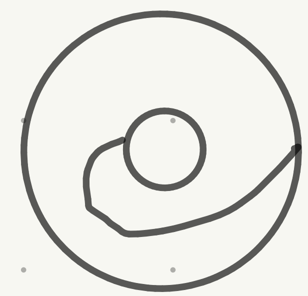
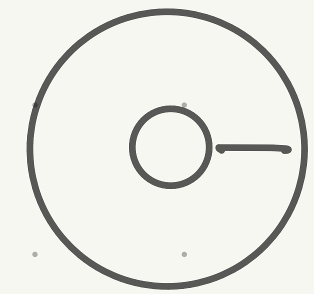


image of $\{(0,s) : s\in I\}$ under $f = f_1$



img for f_t



img for $f_0 = id_{S^1\times I}$

well-defined, cont., $F(\theta, s, 0) = (\theta, s)$ $(id_{S^1\times I})$

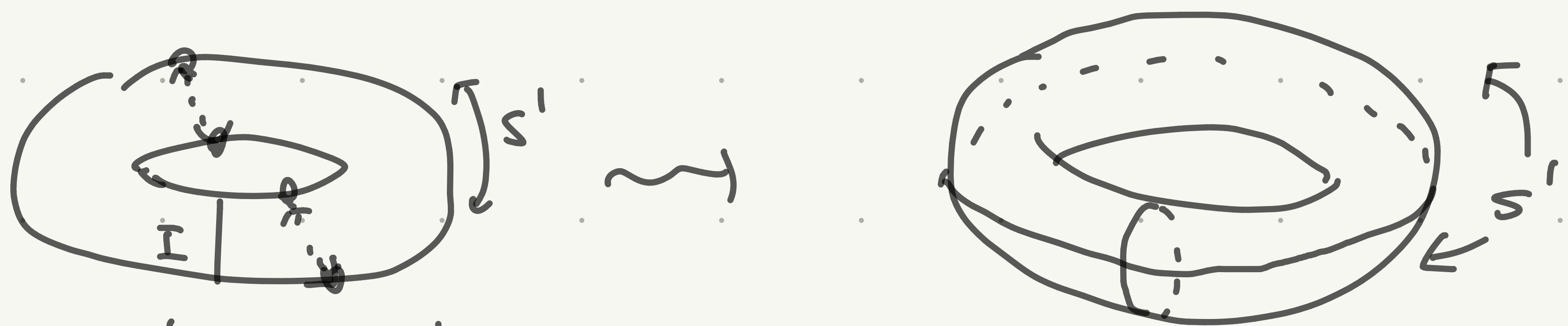
$F(\theta, s, 1) = f(\theta, s)$

always identity on $S^1\times\{1\}$: $F(\theta, 1, t) = (\theta, 1)$

no homotopy ~~between~~ between f and $id_{S^1 \times I}$ that fixes points of $S^1 \times \{0, 1\}$

observation :

- $T = S^1 \times I / (\theta, 0) \sim (\theta, 1)$ "glue the boundaries" is homeomorphic to the torus $S^1 \times S^1$



annulus $S^1 \times I$ $\qquad$ $I/0\sim 1$

- $F : S^1 \times I \times I \to S^1 \times I$ s.t. $F(\theta, i, t) = (\theta, i)$ $i = 0, 1$ would define $T \times I \xrightarrow{F'} T$

Step 1 F' as above is well-defined, continuous

generally the quotient space

$$T = S^1 \times I / (\theta, 0) \sim (\theta, 1)$$

is characterized by $\forall Y$ top. sp.

$f' : T \to Y$ cont. map $\equiv$ $f : S^1 \times I \to Y$ cont. map

s.t. $f(\theta, 0) = f(\theta, 1)$

up to the corresp. $f'(\underline{[(\theta, s)]}) = f(\theta, s)$

$\leadsto$ $F : S^1 \times I \times I \to S^1 \times I$ s.t. $F(\theta, i, t) = (\theta, i)$ $i = 0, 1$

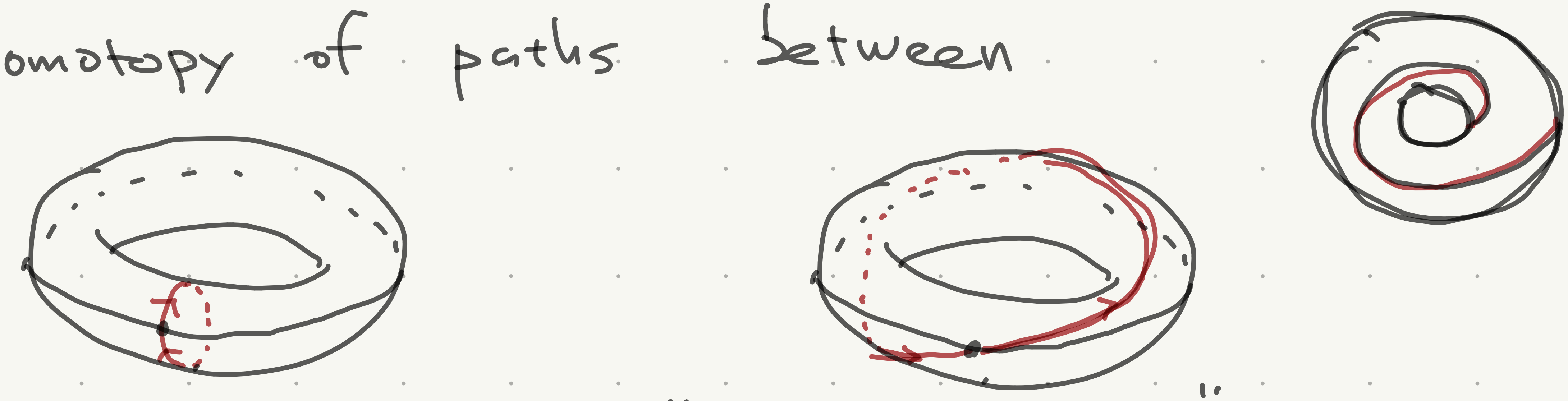
consider $S^1 \times I \times I \xrightarrow{F} S^1 \times I \to T$

defines $T' \to T$ with $[(\theta, 0)] = [(\theta, 1)]$

$$T' = \underbrace{S^1 \times I \times I / (\theta, 0, t) \sim (\theta, 1, t) \quad \forall \theta, t}_{\text{homeomorphic to } T \times I}$$

Step 2 $F' : T \times I \to T$ obtained above gives

homotopy of paths between



following "$\{0\} \times I \subset S^1 \times I$" following "$f(\{0\} \times I) \subset S^1 \times I$"

Step 3 There is no such homotopy

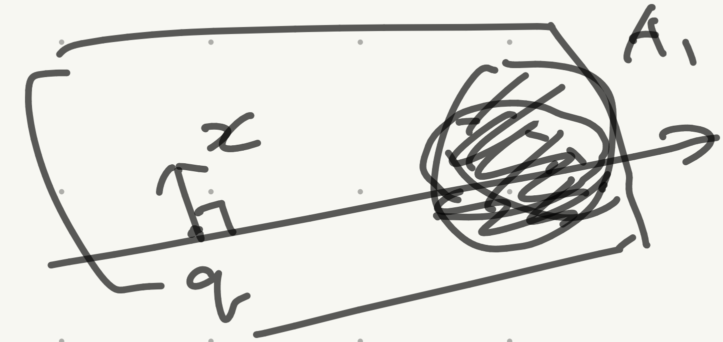
Up to $\pi_1(T, x_0) \cong \mathbb{Z}^2$, the left loop corresponds to $(1,0)$ and the right loop corresponds to $(1,1)$ so they cannot be homotopic

Problem 9

$A_1, A_2, A_3 \subset \mathbb{R}^3$ compact

find a plane $P \subset \mathbb{R}^3$ cutting each A_i to two pieces of equal measure.

Step 1 given any $q \in \mathbb{R}^3$, there is a plane P_q containing q and cutting A_1, A_2 to (respectively) pieces of equal measure



$\because$ given $x \in S^2 \subset \mathbb{R}^3$ define

$f_i(x)$ = volume of the part of A_i, cut by the plane through q, orthogonal to x, in the direction of x

the Borsuk-Ulam th'm gives $x_q \in S^2$ s.t.

$$f_i(x_q) = f_i(-x_q) \qquad (i=1,2)$$

i.e. P_q: plane through q, orth. to x_q

cuts A_1 and A_2 to pieces of same size

Step 2 choose good q so P_q also cuts

A_3 to pieces of same size

choose a line ℓ so that P_q does not

contain ℓ for $q \in \ell$

cuts A_i to equal sizes $i=1,2$

put orientation on ℓ, define

B_+: part of A_3 cut by P_q, on the "pos." side

B_- : part of A_3 cut by P_q, on the "neg." side

$f(q) = (\text{measure of } B_+) - (\text{measure of } B_-)$

then $f(q)$ must change sign at some $q \in \ell$

$\rightsquigarrow$ P_q for this q is the required plane