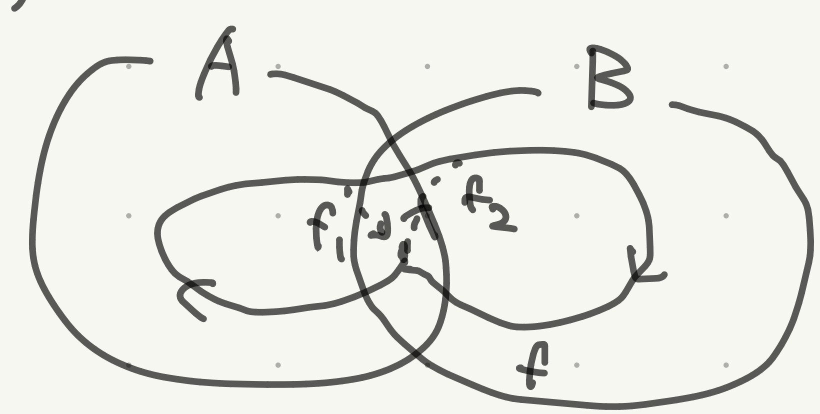
Van Kampen's theorem (§1.2)

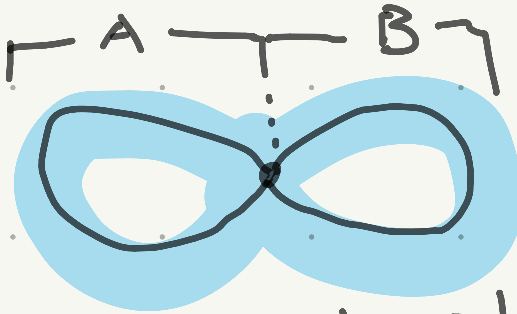
Motivation: suppose $X = A \cup B$, and choose $x_0 \in A \cap B$ $\rightsquigarrow$ how do we compute $\pi_1(X, x_0)$ from $\pi_1(A, x_0)$ and $\pi_1(B, x_0)$?

- if $A \cap B$ is path-connected, any $[f] \in \pi_1(X, x_0)$ can be written as $[f_1] \cdots [f_n]$, with $f_i \in \pi_1(A, x_0)$ or $\pi_1(B, x_0)$



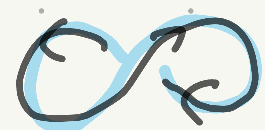
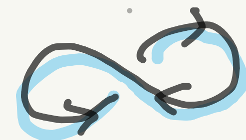
$\rightsquigarrow$ we need to understand the relation between the images of $\pi_1(A, x_0)$ and $\pi_1(B, x_0)$ in $\pi_1(X, x_0)$

"easy" case : $A \cap B \simeq \{x_0\}$ ⇝ there would be no (nontrivial) relation between them



$X = S^1 \vee S^1$

⇝ we should / can distinguish

 vs ,  vs • const, ...

⇝ $\pi_1(X, x_0)$ is going to be "freely" generated by a copy of $\pi_1(A, x_0)$ and a one of $\pi_1(B, x_0)$

more intricate : if $A \cap B$ has nontrivial π_1, the images of $\pi_1(A \cap B, x_0)$ in $\pi_1(A, x_0)$ and $\pi_1(B, x_0)$ should be identified in $\pi_1(X, x_0)$

$$\begin{tikzcd} & A \arrow[dr] & \\ A \cap B \arrow[ur] \arrow[dr] & & X \\ & B \arrow[ur] & \end{tikzcd}$$

Free product of groups

goal: make sense of a group G with subgroups $G_1, G_2, \dots$ such that:

- they generate G (we have enough)
- no nontrivial relation among them (no redundancy)

"trivial rel": $(g_1 g_2)(g_2^{-1} g_1') = g_1 g_1'$, etc. $g_1, g_1' \in G_1$, $g_2 \in G_2$

observation: in particular we should have:

- any elem $g \in G$ can be written as

$$g = g_1 \cdots g_n, \quad g_i \in G_{j_i}$$

for some unique j_i

Def. $(G_\alpha)_{\alpha\in I}$ collection of groups;

- a <u>word</u> (for this collection) is an expression $g_1 g_2 \cdots g_n$ with $g_i \in G_{\alpha_i}$ for some $\alpha_i \in I$ (formally an elem. of $\left(\coprod_{\alpha\in I} G_\alpha\right)^{\{1,\dots,n\}}$ for some n)
- a <u>reduced</u> word is a word $g_1 \cdots g_n$ $(g_i \in G_{\alpha_i})$ such that $\alpha_i \neq \alpha_{i+1}$ for $i = 1, \dots, n-1$, $g_i \neq e$
- the <u>free product</u> of $(G_\alpha)_{\alpha\in I}$ is the set of all reduced words (+ the "empty word" e of length 0) write $\ast_{\alpha\in I} G_\alpha$, $\ast(G_\alpha)_{\alpha\in I}$, etc.

Non-reduced words

$g\,g'\,h \qquad g, g' \in G_\alpha, \quad h \in G_\beta$

$g\,e\,h \qquad g \in G_\alpha \quad e \in G_\beta$ (unit elem)

$h \in G_\gamma$

Reduced $\quad g\,h\,g' \qquad g, g' \in G_\alpha, \quad h \in G_\beta$

Group structure on $*_{\alpha\in I} G_\alpha$

want : - each G_β is a subgroup of $*_\alpha G_\alpha$

- interpret a reduced word $g_1 \cdots g_n \in *_\alpha G_\alpha$ as the product of $g_1, g_2, \cdots, g_n$

formal def of $(g_1 \cdots g_m) \cdot (h_1 \cdots h_n)$

$g_i \in G_{\alpha_i}$ $\quad h_i \in G_{\beta_i}$

- if $\alpha_m \neq \beta_1$, take the reduced word $g_1 \cdots g_m h_1 \cdots h_n$ as the prod.

- if $\alpha_m = \beta_1$ and $g_m \neq h_1^{-1}$, take the reduced word $g_1 \cdots g_{m-1} \underbrace{(g_m h_1)}_{\text{compute in } G_{\alpha_m}} h_2 \cdots h_n$ as the prod.

(cont.) if $\alpha_m = \beta_1$ and $g_m = h_1^{-1}$, look at

$(g_1 \cdots g_{m-1})(h_2 \cdots h_n)$ and use the same

algorithm as above to compute the prod.

(induction on length.; $g_i g_i^{-1} = e$)

empty word

the empty word e is the neutral element

$$(g_1 \cdots g_n) e = g_1 \cdots g_n = e (g_1 \cdots g_n)$$

$\rightsquigarrow$ the reduced word $g_n^{-1} \cdots g_1^{-1}$ becomes the

inverse of $g_1 \cdots g_n$

$$(g_1 \cdots g_n) \cdot (g_n^{-1} \cdots g_1^{-1}) = (g_1 \cdots g_{n-1})(g_{n-1}^{-1} \cdots g_1^{-1}) = \cdots = e$$

Example: <u>free group</u> $\mathbb{F}_S$

S: set (will be the index set "I")

$G_s \cong \mathbb{Z}$ for each $s \in S$, $g_s \in G_s$ corresponding to $1 \in \mathbb{Z}$

$\mathbb{F}_S = \ast_{s \in S} G_s$ free group on the set S

$= \{ g_{s_1}^{n_1} g_{s_2}^{n_2} \cdots g_{s_k}^{n_k} : k = 0, 1, \ldots, s_i \neq s_{i+1}, n_i \in \mathbb{Z} \setminus \{0\} \}$

- generally: group G with generating set $S \subset G$

$\rightsquigarrow$ surjective group hom $\mathbb{F}_S \to G$, $g_s \mapsto s$

$g_{s_1}^{n_1} g_{s_2}^{n_2} \cdots g_{s_k}^{n_k} \mapsto s_1^{n_1} \cdots s_k^{n_k}$

$G = \langle S \mid R \rangle$

generator set

relation

R : some set of words in letters $s \in S$

e.g. $s_1 s_2$, $s_1 s_2 s_1$, ...

$\mathbb{F}_S / \langle R \rangle$

Basic property of free product:

$G_\alpha \xrightarrow{\varphi_\alpha} H$ group hom for each $\alpha \in I$

$\equiv$ group hom $\underset{\alpha \in I}{*} G_\alpha \xrightarrow{\varphi} H$
equivalent

up to the correspondence $\varphi(g) = \varphi_\alpha(g)$ for $g \in G_\alpha$

($\underset{\alpha}{*} G_\alpha$ is the coproduct of $(G_\alpha)_\alpha$ in the category of groups; compare with direct product

$H \xrightarrow{\psi} \prod_\alpha G_\alpha \equiv$ group homs $H \xrightarrow{\psi_\alpha} G_\alpha$

in particular $G \underset{\text{subgrp.}}{>} G_\alpha$, $(G_\alpha)_{\alpha \in I}$ generating G

$\rightsquigarrow$ surjective group hom $\underset{\alpha \in I}{*} G_\alpha \to G$

$$\mathrm{Hom}\left(\mathop{*}_{\alpha\in I} G_\alpha,\ H\right) = \left\{\text{group hom } \mathop{*}_{\alpha\in I} G_\alpha \xrightarrow{\varphi} H\right\}$$

$$\Big\updownarrow\ 1:1$$

$$\prod_{\alpha\in I} \mathrm{Hom}(G_\alpha, H) = \left\{(\varphi_\alpha)_{\alpha\in I} : \varphi_\alpha : G_\alpha \to H\right\}$$

$$\varphi \rightsquigarrow (\varphi_\alpha)_\alpha \qquad \varphi_\alpha = \varphi|_{G_\alpha}$$

$$(\varphi_\alpha)_{\alpha\in I} \rightsquigarrow \varphi(g_1 \cdots g_k) = \varphi_{\alpha_1}(g_1)\cdots\varphi_{\alpha_k}(g_k)$$

$$g_i \in G_{\alpha_i}$$

$h : H \to K$ hom

$$h\circ\varphi \in \mathrm{Hom}\left(\mathop{*}_{\alpha\in I} G_\alpha, K\right)$$

$$\updownarrow$$

$$(h\circ\varphi_\alpha)_{\alpha\in I} \in \prod_{\alpha\in I} \mathrm{Hom}(G_\alpha, K)$$

Examples

1. $\mathbb{Z}_2 * \mathbb{Z}_2 \cong \mathbb{Z} \rtimes_{\pm 1} \mathbb{Z}_2$ (infinite dihedral group) $\qquad \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$

a^{ϵ} $\quad b^{\delta}$ $\quad$ semidirect prod.

$\langle a, b \mid a^2 = e = b^2 \rangle$ $\quad \rightsquigarrow \{(n, g) : n \in \mathbb{Z},\ g \in \mathbb{Z}_2\}$

$\quad \dots$

$$(m, g)\cdot(n, h) = (m + n^g, gh)$$

$n^g = -n$ for the nontriv. g

with nontriv. $g \in \mathbb{Z}_2$: $\left(\begin{array}{l} (1, g)^2 = (1 - 1, g^2) = (0, e) = e \\ (0, g)^2 = (0, g^2) = e \end{array} \right.$

$\rightsquigarrow \mathbb{Z}_2 * \mathbb{Z}_2 \to \mathbb{Z} \rtimes_{\pm 1} \mathbb{Z}_2,\ a \mapsto (1, g),\ b \mapsto (0, g)$

this is an isomorphism

$$\left(b\, ab\, b^{-1} = ba = (ab)^{-1} \right)$$

$\updownarrow$ $\quad (0, g)(1, e)(0, g)^{-1} = (-1, e)$

2 $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z}) / \{\pm I_2\}$ (modular group)

is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_3$

the image of $\begin{array}{c} \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \end{array}$ in $PSL_2(\mathbb{Z})$ has order $\begin{array}{c} 3 \\ 2 \end{array}$

$\rightsquigarrow \mathbb{Z}_2 * \mathbb{Z}_3 \longrightarrow PSL_2(\mathbb{Z})$, which turns out to be an isomorphism

Amalgamated free product

$(G_\alpha)_{\alpha \in I}$: collection of groups

A : group, $\varphi_\alpha : A \to G_\alpha$ homomorphism

(φ_α injective $\rightsquigarrow$ "common subgroup of the G_α")

Def. the <u>amalgamated</u> <u>free</u> <u>product</u> of the groups $(G_\alpha)_\alpha$ over A is

$$\underset{A}{*}(G_\alpha)_{\alpha \in I} = \left(\underset{\alpha \in I}{*} G\right) / N,$$

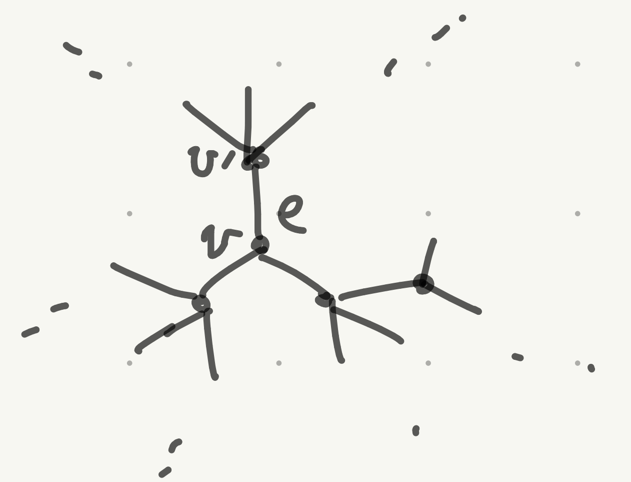
N = the normal subgroup generated by

$$\varphi_\alpha(x)\varphi_\beta(x)^{-1} \quad x \in A, \ \alpha, \beta \in I$$

$\rightsquigarrow$ in $\underset{A}{*}(G_\alpha)_\alpha$ $\quad [\varphi_\alpha(x)] = [\varphi_\beta(x)]$

So $\underset{A}{*} (G_\alpha)_{\alpha \in I}$: generated by $(G_\alpha)_{\alpha \in I}$ - with identification of $\varphi_\alpha(x)$ and $\varphi_\beta(x)$ for $x \in A$ $\alpha, \beta \in I$ but no other nontrivial relation

"Bass - Serre theory"



automorphism groups of trees are built up using amalgamated free product & another operation (HNN extension)

(stabilizer of an edge e) $\subset$ (stab. of a vertex v)

if v is an endpoint of e

Van Kampen's th'm ("usual" version)

Suppose $X = A \cup B$, A & B open,

$A \cap B$ path-connected, $x_0 \in A \cap B$

then $\pi_1(X, x_0) \cong \pi_1(A, x_0) *_{\pi_1(A \cap B, x_0)} \pi_1(B, x_0)$

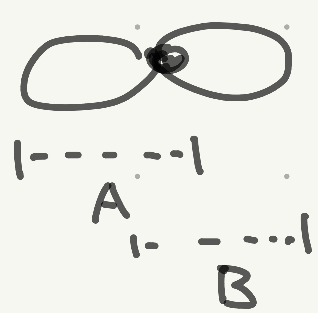
with respect to homs

$$\begin{tikzcd} & \pi_1(A, x_0) \\ \pi_1(A \cap B, x_0) \arrow[ur] \arrow[r] & \pi_1(B, x_0) \end{tikzcd}$$

induced by inclusion $A \cap B \to A$, $A \cap B \to B$ $\qquad \mathbb{Z}_2$

Consequence $\pi_1(S^1 \vee S^1, x_0) \cong \mathbb{F}_2 \; (= \mathbb{Z} * \mathbb{Z})$

$\mathbb{F}_{\{a,b\}}$



and $\pi_1(\underbrace{S^1 \vee \cdots \vee S^1}_{n \times}, x_0) \cong \mathbb{F}_n$

$\pi_1(\vee (X_\alpha)_{\alpha \in I}, x_0) \cong *(\pi_1(X_\alpha, x_0))_{\alpha \in I}$

Van Kampen's th'm (general version)

$X = \bigcup_{\alpha \in I} A_\alpha$, A_α open (path-conn)

$A_\alpha \cap A_\beta \cap A_\gamma$ path-conn. for all $\alpha, \beta, \gamma \in I$

$x_0 \in \bigcap_{\alpha \in I} A_\alpha$

then $\pi_1(X, x_0) \cong \left(\ast_{\alpha \in I} \pi_1(X_\alpha, x_0) \right) / N$

N: the normal subgroup of $\ast_{\alpha \in I} \pi_1(X_\alpha, x_0)$

generated by $i_{\alpha\beta}(\omega)\, i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta, x_0)$

$i_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta, x_0) \to \pi_1(A_\alpha, x_0)$

$i_{\beta\alpha} : \pi_1(A_\beta \cap A_\alpha, x_0) \to \pi_1(A_\beta, x_0)$

i.e. $[i_{\alpha\beta}(\omega)] = [i_{\beta\alpha}(\omega)]$ $(\omega \in \pi_1(A_\alpha \cap A_\beta))$ in $\pi_1(X)$