Van Kampen's theorem, part 2

General claim: $X = \bigcup_{\alpha \in I} A_\alpha$, A_α open, s.t.

- $A_\alpha \cap A_\beta \cap A_\gamma$ path connected for $\forall \alpha, \beta, \gamma \in I$

  (A_α, $A_\alpha \cap A_\beta$ also)

- $x_0 \in \bigcap_{\alpha \in I} A_\alpha$

then $\pi_1(X, x_0) \cong \left(\ast_{\alpha \in I} \pi_1(A_\alpha, x_0) \right) / N$

N: normal subgrp generated by $i_{\alpha\beta}([\omega]) \, i_{\beta\alpha}([\omega])^{-1}$

for $[\omega] \in \pi_1(A_\alpha \cap A_\beta, x_0) \xrightarrow{i_{\alpha\beta}} \pi_1(A_\alpha, x_0)$

Particular case ($I = \{1, 2\}$): $X = A_1 \cup A_2$

$A_1 \cap A_2$ path-conn., $x_0 \in A_1 \cap A_2$

$\Rightarrow \pi_1(X, x_0) = \pi_1(A_1, x_0) \ast_{\pi_1(A_1 \cap A_2, x_0)} \pi_1(A_2, x_0)$

We already know: $\ast_{\alpha \in I} \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$ is surjective $\iff$ any $[f] \in \pi_1(X, x_0)$ can be written as $\underbrace{[f] = [f_1] \cdots [f_k]}$, $[f_i] \in \pi_1(A_{\alpha_i}, x_0)$

call this a <u>factorization</u> of $[f]$

we say two factorizations $[f] = [f_1] \cdots [f_k]$ and $[f] = [f_1'] \cdots [f_\ell']$ are <u>equivalent</u> if they are related by two types of transformations:

- if f_j and f_{j+1} are both loops in A_{α_j},

$$[f_1] \cdots [f_k] \rightsquigarrow [f_1] \cdots [f_j \cdot f_{j+1}] \cdots [f_k]$$

- if f_j is a loop in $A_{\alpha_j} \cap A_\beta$, interpret $[f_j]$ as an elem. in $\pi_1(A_\beta, x_0)$ instead of $\pi_1(A_{\alpha_j}, x_0)$

the kernel of $\ast_{\alpha \in I} \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$ is generated

by $i_{\alpha\beta}([\omega]) \, i_{\beta\alpha}([\omega])^{-1}$ for $[\omega] \in \pi_1(A_\alpha \cap A_\beta, x_0)$

$\Leftrightarrow$ any two factorizations of $[f]$ are equivalent

$\rightsquigarrow$ we will check the latter claim

Take two factorizctions

$$[f] = [f_1] \cdots [f_k] \quad \text{and} \quad [f] = [f_1'] \cdots [f_\ell']$$

$\rightsquigarrow$ $\exists$ homotopy of paths $F : I \times I \to X$ implementing

$$f_1 \cdot \cdots \cdot f_k \simeq f_1' \cdot \cdots \cdot f_\ell'$$

Step 1 find partitions $0 = s_0 \leq \cdots \leq s_m = 1$,

$0 = t_0 \leq \cdots \leq t_n = 1$ s.t.

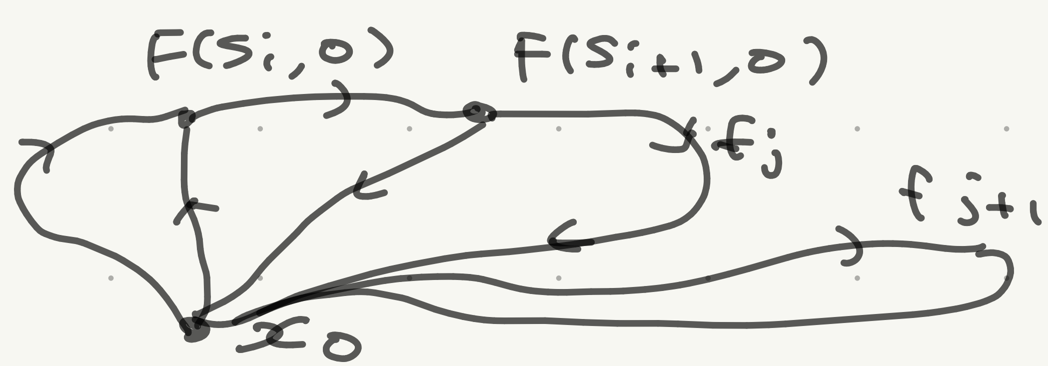
- each small rectangle $[s_i, s_{i+1}] \times [t_j, t_{j+1}]$ is mapped into some A_α by F

  (use compactness)

- at $t = 0$: the loops

  $(x_0 \rightsquigarrow F(s_i, 0)) \cdot F(\sim, 0)|_{[s_i, s_{i+1}]} \cdot (F(s_{i+1}, 0) \to x_0)$

  form a factorization equiv. to $[f] = [f_1] \cdots [f_k]$



- at $t = 1$: same for $[f] = [f_1'] \cdots [f_\ell']$

Step 2 modify and reduce to "smaller steps"

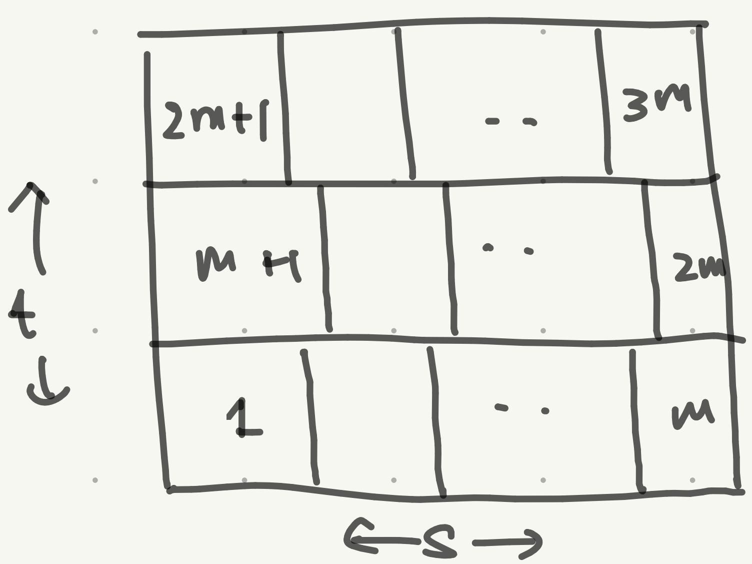
- assume there are only three "rows" (t-direction)

  | we can connect homotopy / equivalences

  $[f_1]\cdots[f_k] \leftrightarrow [f'_1]\cdots[f'_\ell] \leftrightarrow [f''_1]\cdots[f''_m]$

- move vertical lines in the middle row so any point is touching at most three rectangles

  still satisfying same assumptions



← m x rectangles

R_i : i-th rectangle in the above config

$A_i = A_{\alpha_i}$: contains $F(R_i)$; choose $A_1, \ldots, A_m, A_{2m+1}, \ldots, A_{3m}$

acc. to factoriz. f_i, f'_j

Step 3 enumerate paths $\gamma_0, \cdots, \gamma_{3m}$ in $I \times I$

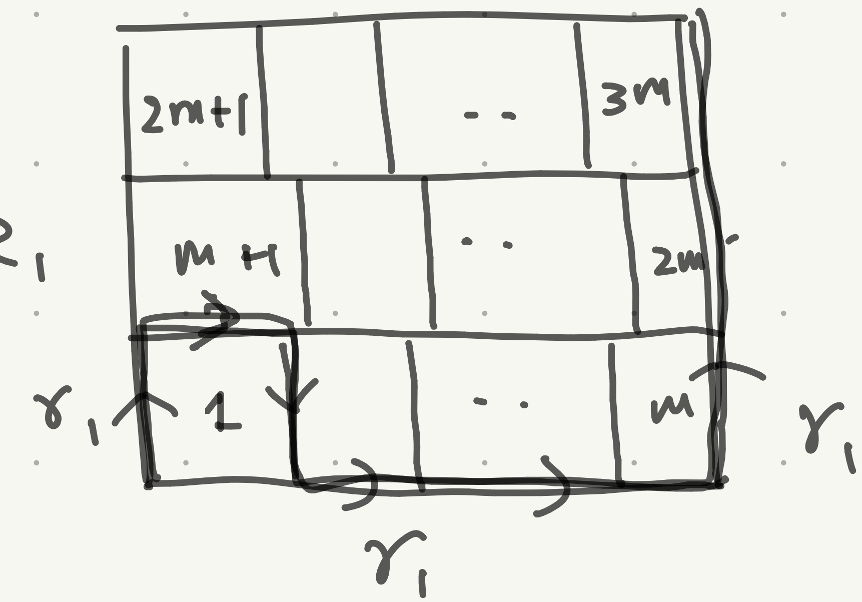
behind deform. of loops $f_1 \circ \cdots \circ f_k \rightsquigarrow f_1' \circ \cdots \circ f_\ell'$

- γ_0 = go the bottom edge, then the right

  $F|_{\gamma_0} = f_1 \circ \cdots \circ f_k$

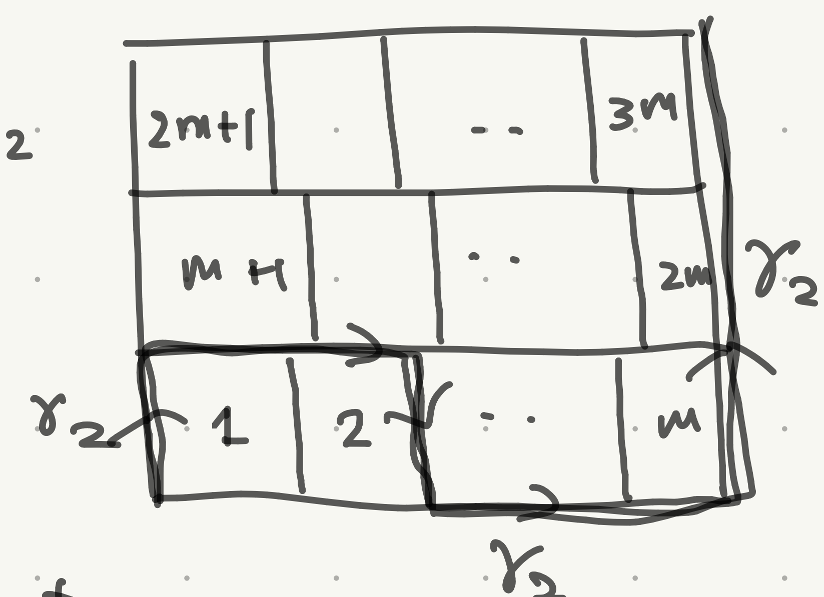
- γ_1 : the path that goes around R_1

  then go like γ_0



- γ_2 : path that goes around R_1, R_2

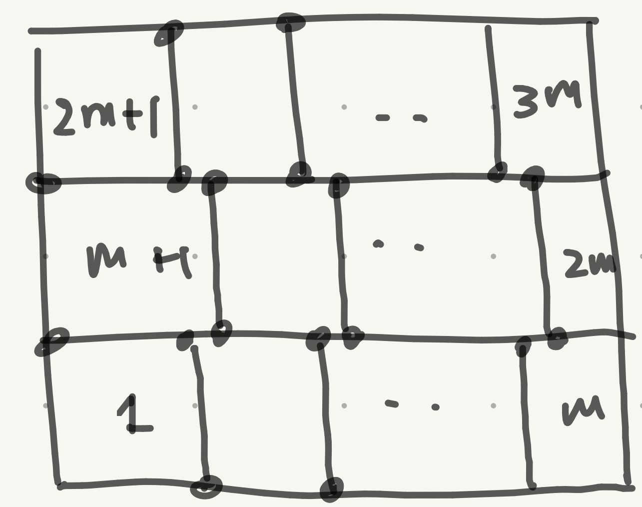
  then go like γ_1



- γ_{3m} : go up the left edge, then the top

  $F|_{\gamma_{3m}} = f_1' \circ \cdots \circ f_\ell'$

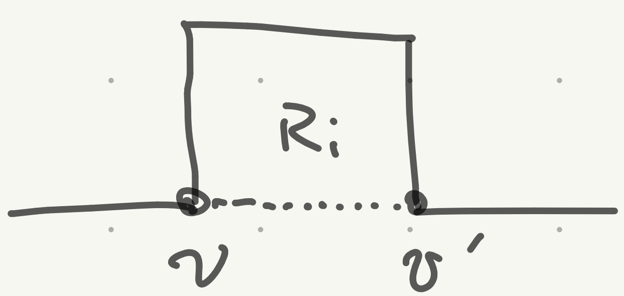
Step 4 $\quad Fl_{\gamma_i}$ has equivalent factorization as $Fl_{\gamma_{i-1}}$



at each "vertex" v of tesselation take adjacent rectangles R_i, R_j, R_k choose a path $g_v : x_0 \sim v$ in $A_i \cap A_j \cap A_k$

- suppose we have constructed factorization for $Fl_{\gamma_{i-1}}$

$\rightsquigarrow$ when γ_i goes around R_i we get a loop in A_i by composing g_v, Fledges on R_i, $\overline{g_{v'}}$



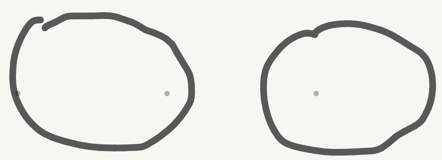
$\rightsquigarrow$ - factorization of Fl_{γ_i}

- equiv. to that of $Fl_{\gamma_{i-1}}$

Examples

1. L : link of two circles in $\mathbb{R}^3$ $\rightsquigarrow$ $\pi_1(\mathbb{R}^3 \setminus L, x_0)$

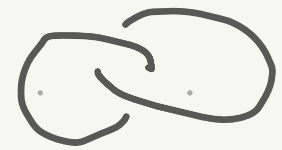
$\uparrow$ any pt outside L



$\cdot x_0$

A B

or



$\cdot x_0$

A B

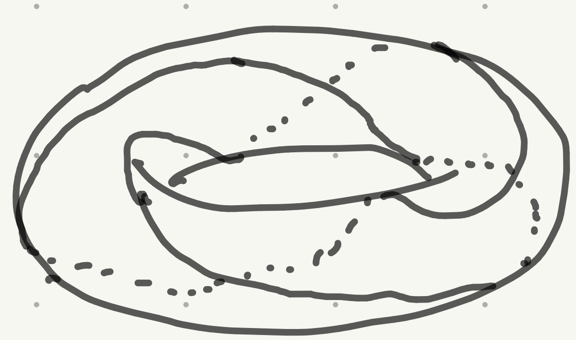
$\rightsquigarrow \mathbb{Z} * \mathbb{Z}$ $\qquad$ $\mathbb{Z} \times \mathbb{Z}$

2. $\pi_1(\mathbb{R}^3 \setminus K_{m,n}, x_0)$ for $K_{m,n} \subset S^1 \times S^1 \subset \mathbb{R}^3$:

torus knot $\qquad$ "usual" torus surface

image of $S^1 \to S^1 \times S^1$, $z \mapsto (z^m, z^n)$

$K_{2,3}$



(m, n : coprime)

$\langle a, b : a^m = b^n \rangle$

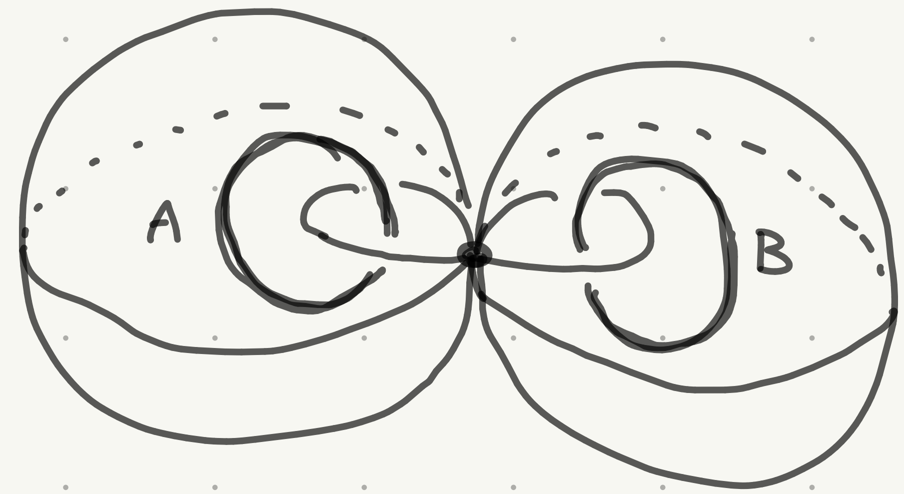
$\pi_1(\mathbb{R}^3 \setminus L, x_0) \cong \mathbb{Z} * \mathbb{Z}$ for "unlink" L: 

A B

key step: $\mathbb{R}^3 \setminus L$ contains $(S^1 \vee S^2) \vee (S^1 \vee S^2)$

as deformation retract

⇝ same π_1



Van Kampen's th'm gives

$$\pi_1((S^1 \vee S^2) \vee (S^1 \vee S^2), *) \cong \pi_1(S^1 \vee S^2, *) * \pi_1(S^1 \vee S^2, *)$$

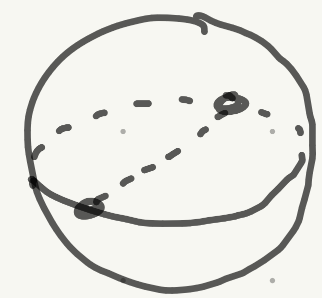
basepoint free prod

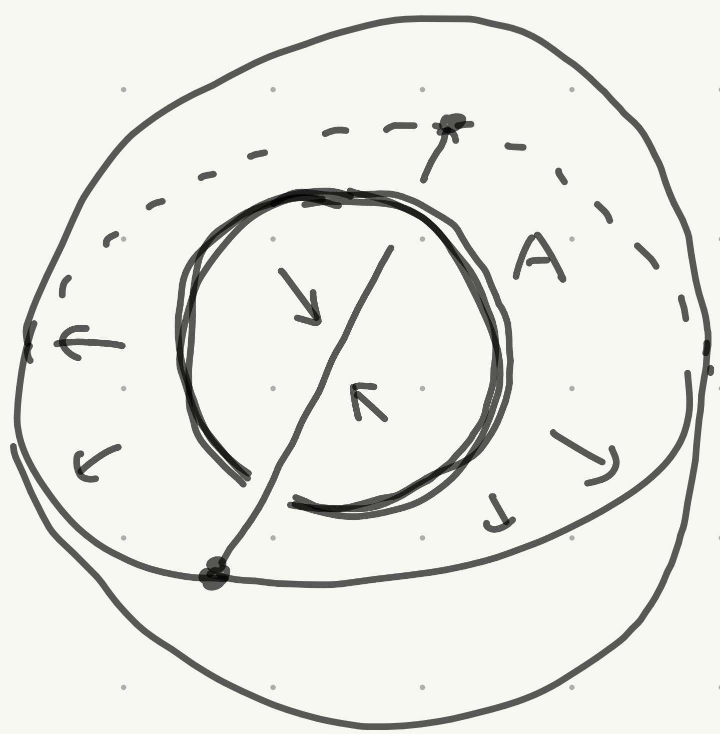
$$\cong \Big(\underbrace{\pi_1(S^1, *)}_{\cong \mathbb{Z}} * \underbrace{\pi_1(S^2, *)}_{\text{triv.}}\Big)^{*2} = \mathbb{Z} * \mathbb{Z}$$

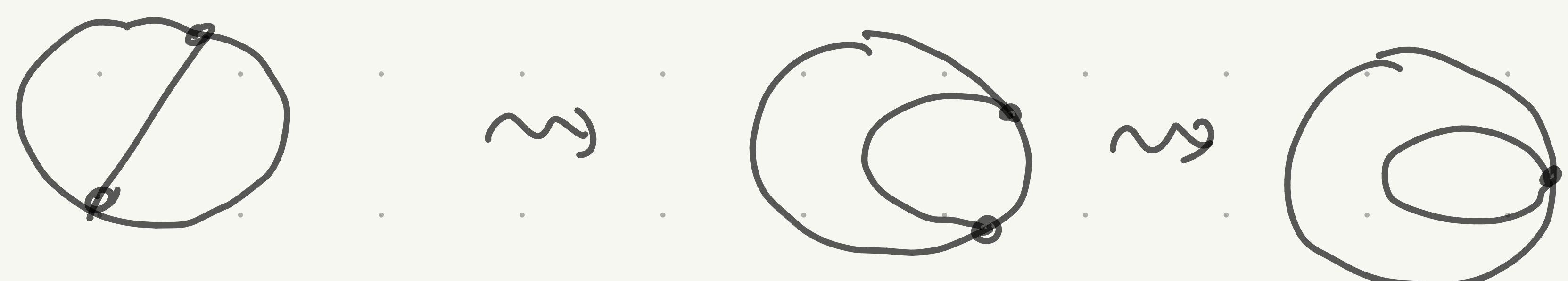
how to see $(S^1 \vee S^2) \vee (S^1 \vee S^2) \subset \mathbb{R}^3 \setminus (A \cup B)$.

unlinked circles

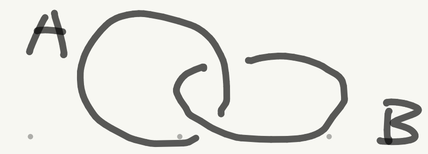
want : $S^1 \vee S^2 \simeq_{d.r.} \mathbb{R}^3 \setminus A$

easier (but inefficient) :  $\subset \mathbb{R}^3 \setminus A$ def. retr.



then deform 

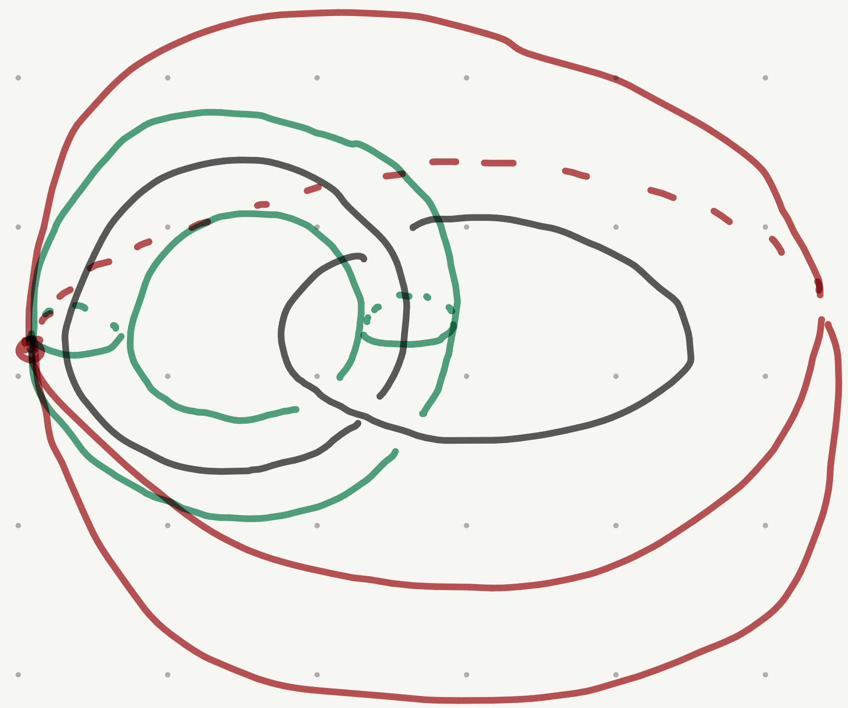
inside S^2

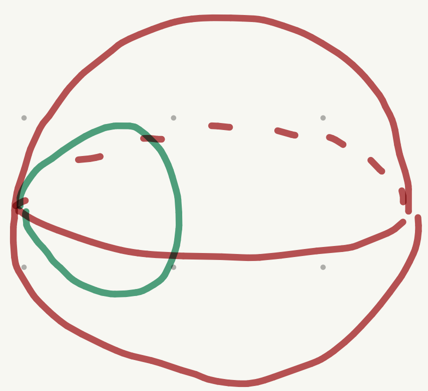
$\pi_1(\mathbb{R}^3 \setminus L, x_0) \cong \mathbb{Z} \times \mathbb{Z}$ for L : 

key step : $\mathbb{R}^3 \setminus L$ contains $(S^1 \times S^1) \vee S^2$

as deformation retract

$\rightsquigarrow$ same π_1





$S^2 \vee S^1 \simeq \mathbb{R}^3 \setminus B$

Van Kampen's th'm gives

$$\pi_1((S^1 \times S^1) \vee S^2, *) \cong \underbrace{\pi_1(S^1 \times S^1, *)}_{\mathbb{Z} \times \mathbb{Z}} * \underbrace{\pi_1(S^2, *)}_{\text{triv.}}$$

$\pi_1(\mathbb{R}^3 \setminus K_{m,n}, x_0) \cong \langle a, b : a^m = b^n \rangle$ for torus knots

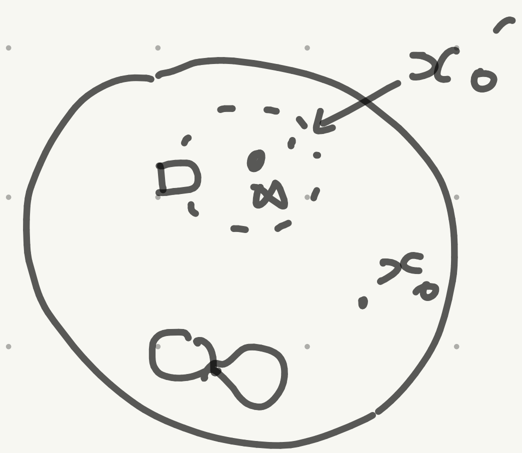
Step 1 $\quad \pi_1(\mathbb{R}^3 \setminus K_{m,n}, x_0) \cong \pi_1(S^3 \setminus K_{m,n}, x_0)$ for $S^3 = \mathbb{R}^3 \cup \{\infty\}$

$S^3 = \mathbb{R}^3 \cup D$ with open neighborhood D of ∞

s.t. $D \cong$ open 3-ball, $\mathbb{R}^3 \cap D \cong$ open 3-ball $\setminus \{pt\}$

$\simeq S^2$
($\simeq$ homotopy equiv)

$K_{m,n} \cap D = \emptyset$; like



$\rightsquigarrow$ Van Kampen implies

$$\pi_1(S^3 \setminus K_{m,n}, x_0') \cong \pi_1(\mathbb{R}^3 \setminus K_{m,n}, x_0') \ast_{\pi_1(D \cap \mathbb{R}^3, x_0')} \pi_1(D, x_0')$$

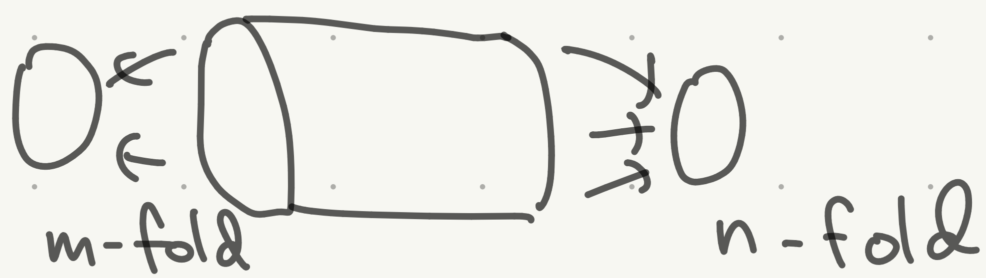
$\cong \pi_1(\mathbb{R}^3 \setminus K_{m,n}, x_0)$

Step 2 finding a 2-dim. cell complex X

embedding $X \subset S^3 \setminus K_{m,n}$ as deform. retract

- construction of X:

$$X = (S^1 \times I) \Big/ \begin{array}{l} (z,0) \sim (e^{\frac{2\pi i}{m}} z, 0) \\ (z,1) \sim (e^{\frac{2\pi i}{n}} z, 1) \end{array} \quad \text{for } z \in S^1 \subset \mathbb{C}$$



- $S^3 \cong (S^1 \times D^2) \amalg (D^2 \times S^1) \Big/$ glue over boundaries $S^1 \times S^1$

contains $K_{m,n}$

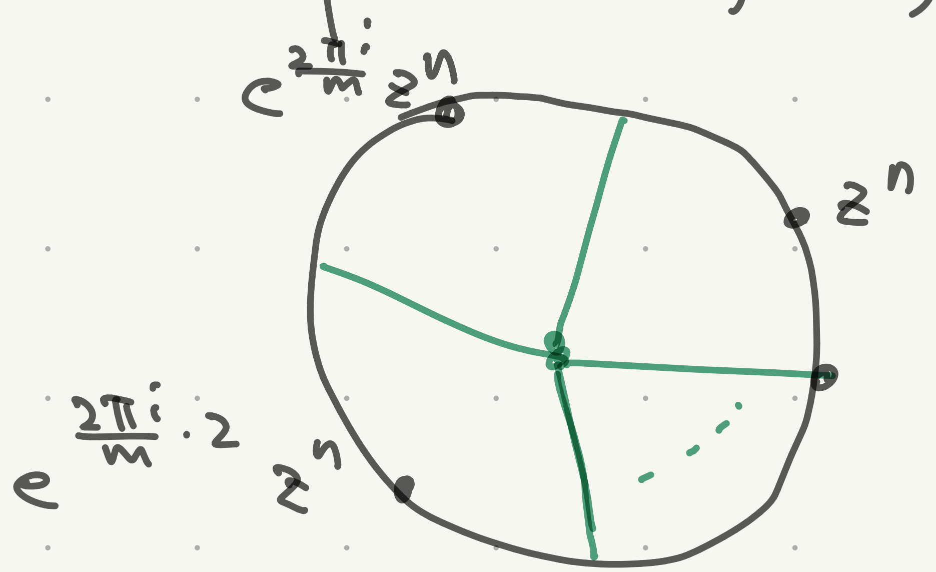
embedding $X \hookrightarrow S^3 \setminus K_{m,n}$

embed the img of $S^1 \times [0, \frac{1}{2}]$ to $S^1 \times D^2$

$S^1 \times [\frac{1}{2}, 1]$ to $D^2 \times S^1$

in $S^1 \times D^2$: $K_{m,n}$ looks like the trajectory of

m points going around $S^1 = \partial D^2$ at n-times



the speed of going around

the first factor S^1

$\rightsquigarrow$ deform retr. to a mapping

cone for $S^1 \to S^1$, $z \mapsto z^m$

$\left(S^1 \times [0, \frac{1}{2}] \,/\, (z, 0) \sim (e^{\frac{2\pi i}{m}} z, 0) \right)$

$(\{w\} \times D^2) \cap K_{m,n}$

$w = z^m$

Step 3 computing $\pi_1(X, *)$

A : img of $S^1 \times [0, \frac{1}{2}]$

B : img of $S^1 \times [\frac{1}{2}, 1]$

$\Rightarrow A \cap B \cong S^1 \ni *$

$\rightsquigarrow$ Van Kampen's thm gives

$$\pi_1(X, *) \cong \pi_1(A, *) *_{\pi_1(A\cap B, *)} \pi_1(B, *)$$

obs : A deform. retracts to the img of $S^1 \times \{0\}$

$$\rightarrow \pi_1(A, *) \cong \pi_1(A, \underbrace{*'}_{\text{from } S^1 \times \{0\}}) \cong \pi_1(S^1, *') \cong \mathbb{Z}$$

(Same with B)

$\pi_1(A \cap B, *) \to \pi_1(A, *)$ is $\mathbb{Z} \to \mathbb{Z}$, $x \mapsto mx$

so $\underbrace{\pi_1(A, *)}_{\langle a \rangle} *_{\pi_1(A\cap B, *)} \underbrace{\pi_1(B, *)}_{\langle b \rangle} = \langle a, b : \underbrace{a^m = b^n}_{\text{img of } 1 \in \mathbb{Z} \cong \pi_1(A\cap B, *)} \rangle$