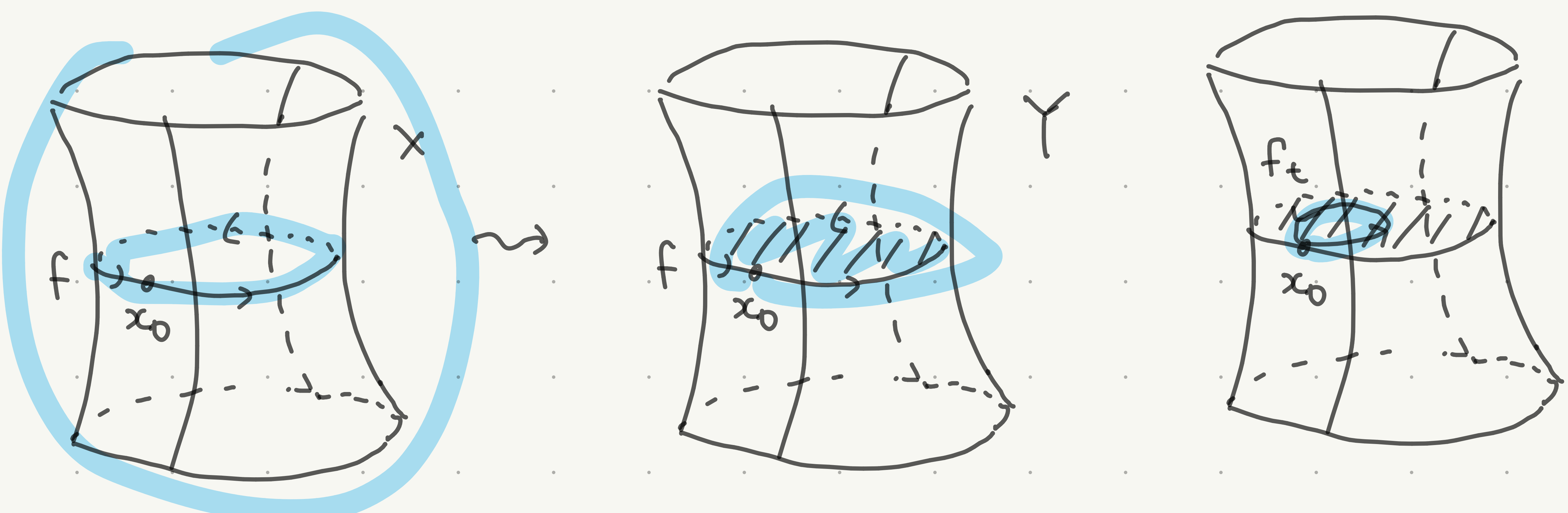
Fundamental group of (2-dimensional) cell complexes

general idea : $f : \underset{\mathbb{C}}{\underset{\cap}{S^1}} \to X$ loop based at x_0 $(= f(1))$

when we glue D^2 using f :

$$Y = (X \amalg D^2)/ z \sim f(z) \quad \text{for } z \in S^1 = \partial D^2$$

f becomes contractible in Y



(cont.) $\pi_1(X, x_0) \to \pi_1(Y, x_0)$ sends $[f]$ to e

induced by $X \hookrightarrow Y$

i.e. we can "kill" any element of $\pi_1(X, x_0)$ by (add some relation on $\pi_1(X, x_0)$)

attaching a 2-cell to X

Q 1 can $\pi_1(Y, x_0)$ be "bigger" by this process?

Q 2 what happens when we attach some n-cell with $n > 2$? : $(X \amalg D^n) / \underset{\text{for } x \in S^{n-1}}{x \sim f(x)}$ for $f : S^{n-1} \to X$

Rem : attaching a 1-cell can make π_1 bigger

Prop (1.26. (a))

X topological space, $x_0 \in X$

$\varphi_\alpha : S^1 \to X$ $(\alpha \in I)$ cont. maps

$Y = X \amalg (\coprod_{\alpha \in I} D^2) / z \sim \varphi_\alpha(z)$ for $z \in S^1 = \partial D^2$ at α-th component

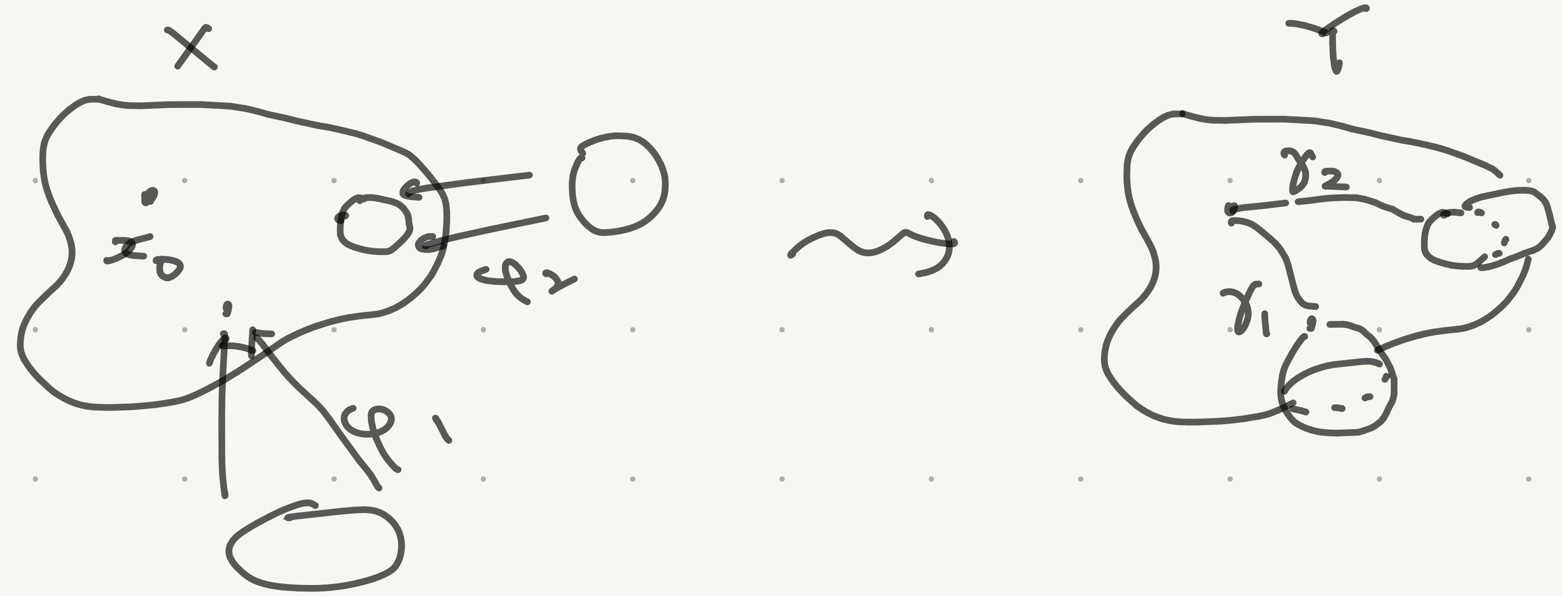
then the hom. $\pi_1(X, x_0) \to \pi_1(Y, x_0)$ induced by $X \hookrightarrow Y$ is surjective

kernel N: the normal subgroup generated by

$\underbrace{\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma}_\alpha}_{\text{loop based at } x}$; $\gamma_\alpha : I \to X$, $\gamma_\alpha(0) = x_0$

flipped path

$\gamma_\alpha(1) = \varphi_\alpha(1)$

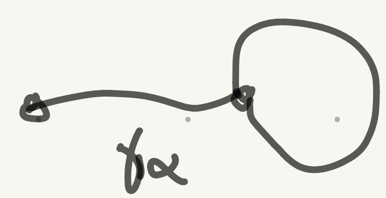


Rough idea: $Y = X \cup \left(\bigcup_{\alpha \in I} \text{img of } D^2 \cup \gamma_\alpha \right)$

$\overset{\text{Van Kampen}}{\Longrightarrow}$ $\pi_1(Y, x_0) \triangleq$ quot. of

$$\pi_1(X, x_0) * \left(\ast_{\alpha \in I} \underbrace{\pi_1(D^2 \cup \text{img}\,\gamma_\alpha, x_0)}_{\text{trivial}} \right)$$

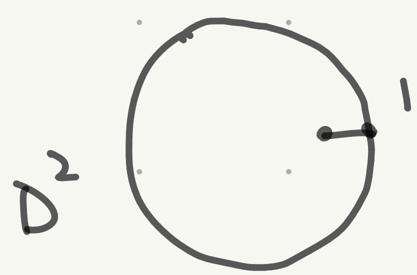
kernel from intersections



img of S^1

to make this precise: we need open cover "thicken" the images of γ_α

- fix a segment in D^2 attached to $1 \in S^1$



$\psi : I \to D^2$, $0 \mapsto 1$

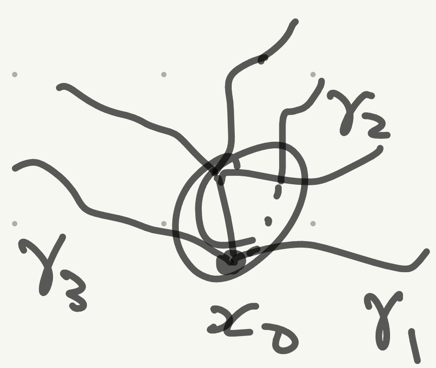
- take $Z = X \sqcup \left(\coprod_{\alpha \in I} D^2 \sqcup I^2 \right) /$

$z \sim \varphi_\alpha(z)$ for $z \in S^1 = \partial D^2$

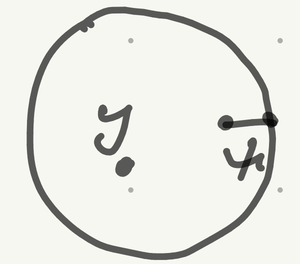
$(s, 0) \sim \gamma_\alpha(s)$ for $s \in I$

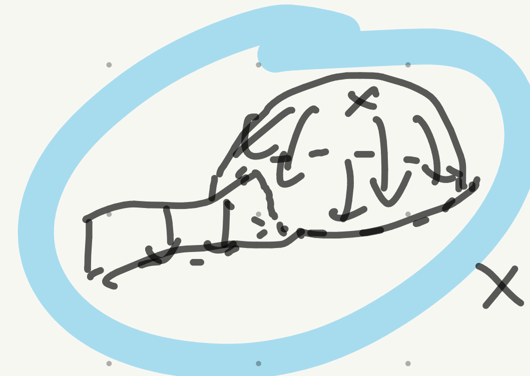
$(1, t) \sim \psi(t)$

at α-th component

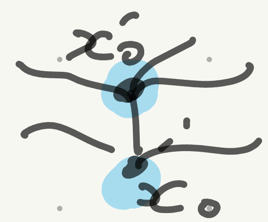


all copies of $\{0\} \times I$ identified open cover of Z :

- choose y not in the img of φ_1 D^2 

- $A = Z \setminus \left(\bigcup_{\alpha \in I} \{y\} \text{ at } \alpha\text{-th component} \right)$ 

- $B = Z \setminus X$ = union of  (glued at the end of strips)

- x_0' the point "above" x_0 

$x_0' \in A \cap B$

$A \supset X$ deformation retract $\pi_1(A, x_0') \cong \pi_1(X, x_0)$

B contractible $\pi_1(B, x_0') = \{e\}$

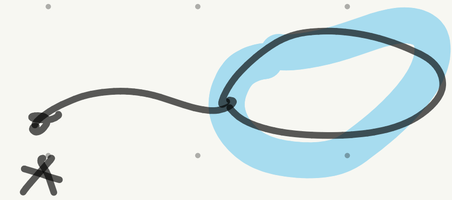
$Z \supset Y$ deformation retract $\pi_1(Z, x_0') \cong \pi_1(Y, x_0)$

Van Kampen: $\underbrace{\pi_1(Z, x_0')}_{\pi_1(Y, x_0)} \cong \pi_1(A, x_0') *_{\pi_1(A\cap B, x_0')} \pi_1(B, x_0')$

$\cong \underbrace{\pi_1(A, x_0')}_{\pi_1(X, x_0)} / \langle \text{img of } \pi_1(A\cap B, x_0') \rangle$

$A\cap B$: union of  for $\alpha \in I$

(glued at the end of strips)

$\simeq$ union of 

for $\alpha \in I$ glued at *

$\leadsto \pi_1(A\cap B, x_0') = \langle [\gamma_\alpha \cdot \varphi_\alpha \cdot \bar{\gamma}_\alpha] : \alpha \in I \rangle$

up to $x_0 \sim x_0'$

Prop (1.26. (b))

X topological space, $x_0 \in X$

$\varphi_\alpha : S^{n-1} \to X$ $(\alpha \in I)$ cont. maps $(n > 2)$

$Y = X \amalg (\coprod_{\alpha \in I} D^n) / z \sim \varphi_\alpha(z)$ for $z \in S^{n-1} = \partial D^n$ at α-th component

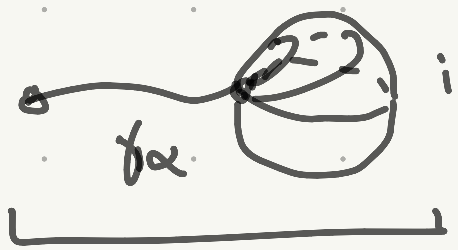
then the hom. $\pi_1(X, x_0) \to \pi_1(Y, x_0)$ induced by $X \hookrightarrow Y$ is an isomorphism

Rough idea: $Y = X \cup \left(\bigcup_{\alpha \in I} \text{img of } D^n \cup \gamma_\alpha \right)$

$\underset{\text{Van Kampen}}{\Longrightarrow}$ $\pi_1(Y, x_0) \triangleq$ quot. of

$$\pi_1(X, x_0) * \left(\mathop{\ast}_{\alpha \in I} \pi_1 \underbrace{(D^n \cup \text{img}\, \gamma_\alpha, x_0)}_{\text{trivial}} \right)$$

kernel from intersections



img of S^{n-1}

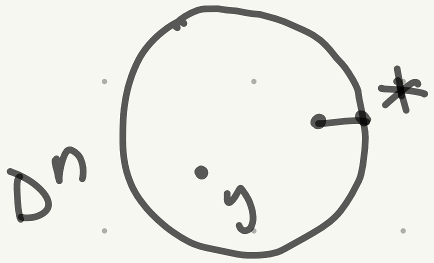
trivial fundamental group

Again we need to thicken γ_α so we have an open covering $Z = A \cup B$

$\cup$ def. retr.

Y

• fix a segment in D^n attached to $* \in S^{n-1}$



$\varphi_1 : I \to D^n$, $0 \mapsto *$

$\in S^{n-1} \subset D^n$

• take $Z = X \sqcup \left(\bigsqcup_{\alpha \in I} D^n \sqcup I^2 \right) \Big/$

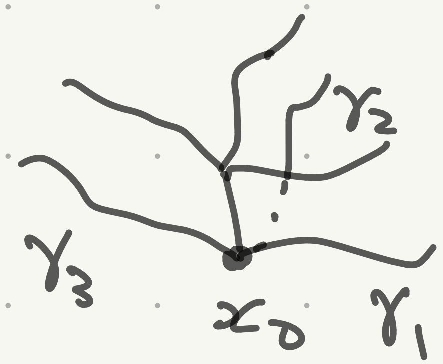
$z \sim \varphi_\alpha(z)$, $z \in S^{n-1} = \partial D^n$

$(s, 0) \sim \gamma_\alpha(s)$, $s \in I$

$(1, t) \sim \varphi_1(t)$

at α-th component

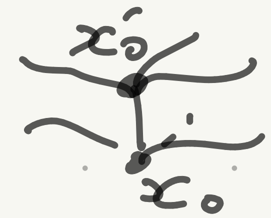
all copies of $\{0\} \times I$ identified



• choose y not in the img of φ_1

$\rightsquigarrow A = Z \setminus \left(\bigcup_{\alpha \in I} \{y\} \text{ at } \alpha\text{-th component} \right)$

• $B = Z \setminus X$ = union of  (glued at the end of strips)

- x_0' the point "above" x_0 

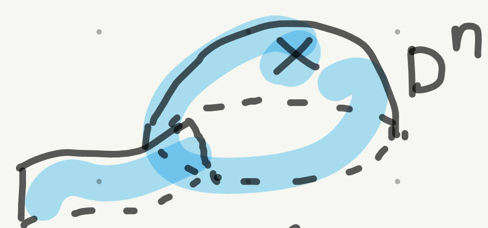
$x_0' \in A \cap B$

we have:

$A \supset X$ deformation retract $\pi_1(A, x_0') \cong \pi_1(X, x_0)$

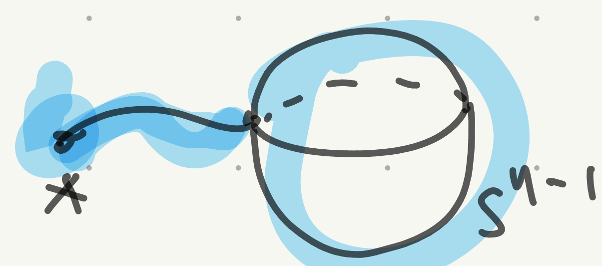
B contractible $\pi_1(B, x_0') = \{e\}$

$Z \supset Y$ deformation retract $\pi_1(Z, x_0') \cong \pi_1(Y, x_0)$

$A \cap B$: union of 

for $\alpha \in I$

(glued at the end of strips)

$\simeq$ union of 

for $\alpha \in I$ glued at $*$

trivial π_1

Van Kampen's th'm implies

$$\pi_1(Z, x_0') \cong \frac{\underbrace{\pi_1(A, x_0')}_{\substack{\cong \\ \pi_1(X, x_0)}} * \pi_1(B, x_0')}{\pi_1(A \cap B, x_0')}$$

$$\left(\begin{array}{c} \cong \\ \pi_1(Y, x_0) \end{array} \right)$$

trivial

$\Rightarrow \pi_1(X, x_0) \cong \pi_1(Y, x_0)$

Prop (1.26. (c)) X : cell complex X^k : k-skeleton

$x_0 \in X^2$

then $X^2 \hookrightarrow X$ induces an isom $\pi_1(X^2, x_0) \xrightarrow{\sim} \pi_1(X, x_0)$

Key Lem (A.1) $C \subset X$ cpt $\Rightarrow C \subset X^k$ for some k

$\rightsquigarrow$ (any class of $\pi_1(X, x_0)$ comes from $\pi_1(X^k, x_0) \cong \pi_1(X^2, x_0)$.

any homotopy of loops happen in X^k for some k

Proof of Key Lem.

Recall : the topology of X is characterized by

$F \subset X$ closed $\Longleftrightarrow$ $\forall k$ $F \cap X^k$ closed in X^k

$U \subset X$ open $\Longleftrightarrow$ $\forall k$ $U \cap X^k$ open in X^k

Suppose $C \subset X$ compact is not contained in any X^k

Take $x_k \in C \setminus X^{k-1}$, $S := \{x_1, x_2, \cdots\} \subset C$.

S is closed in X ($\Rightarrow$ cpt) :

$S \cap X^k = \{x_1, \cdots, x_k\}$ closed in X^k

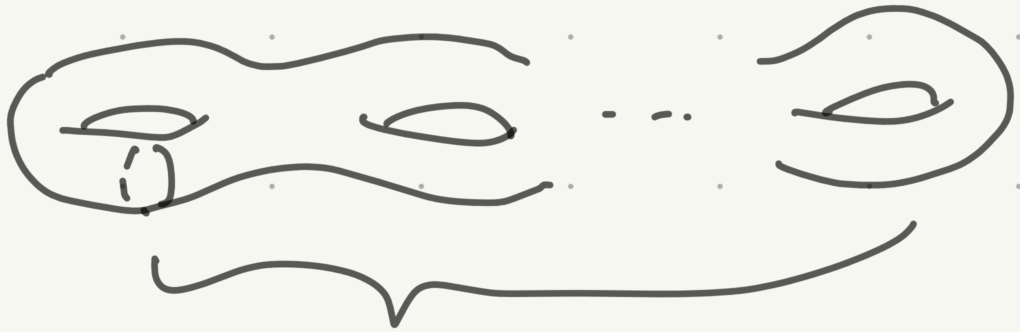
S is discrete for the induced topology

$S \setminus \{x_k\}$ is closed for any k

contra.

Fundamental group of closed oriented surfaces

M_g :

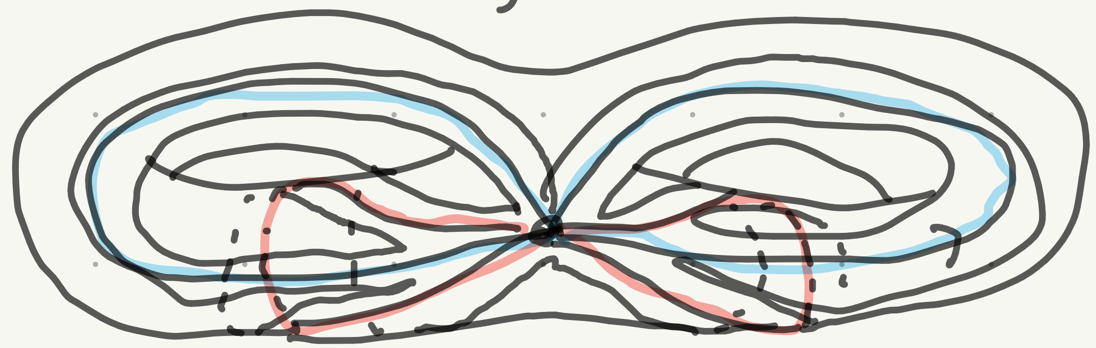


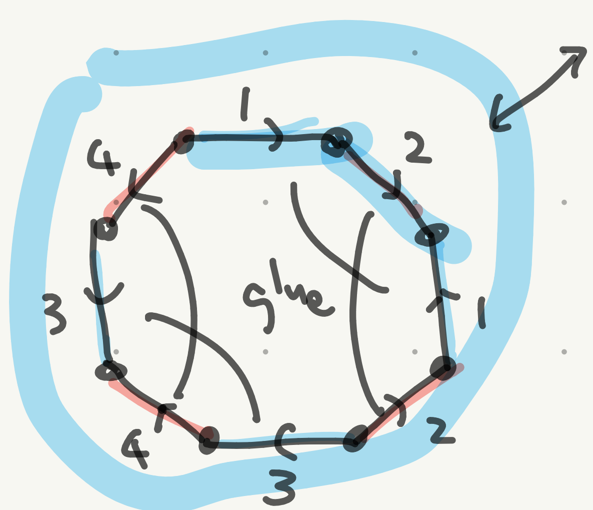
$g \times$ holes

Recall : 2-dim cell complex structure on M_g

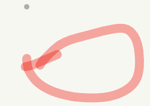
- one 0-cell
- $2g \times$ 1-cell
- one 2-cell

$g = 2$





 "longitude"

 "meridian"

in this cell complex structure:

$X^1 = S^1 \vee \cdots \vee S^1$ $2g$-times

$\pi_1(X^1, *) = \langle a_1, b_1, \ldots, a_g, b_g \rangle \cong \mathbb{F}_{2g}$

"longitude" meridian

$M_g = X^2 = (X^1 \amalg D^2)/$ glueing

boundary of D^2 given by

$\underbrace{a_1 b_1 a_1^{-1} b_1^{-1}}_{\text{commutator}} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$

$[a_1, b_1]$

$\pi_1(X^2, *) = \langle a_1, b_1, \ldots, a_g, b_g :$ above expression $= e \rangle$

Lem $\pi_1(M_g, *) \not\cong \pi_1(M_h, *)$ for $g \neq h$

the abelianization of $\pi_1(M_g, *) \cong (\mathbb{F}_{2g})^{ab} \cong \mathbb{Z}^{2g}$

$$G^{ab} = G/\underbrace{[G, G]}_{\text{commutator subgroup}}$$

commutator subgroup

(normal) subgroup generated by

$$[x, y] = xyx^{-1}y^{-1} \quad (x, y \in G)$$

Cor $M_g \not\cong M_h$ if $g \neq h$

Realizing arbitrary group as π_1

G : group

$S \subset G$: generating set

$R \subset \mathbb{F}_S$: set of "relations" $\nearrow$ prescribing words in generators that should be trivial in G

So $G \cong \mathbb{F}_S / \langle R \rangle$ up to the surjective

hom $\mathbb{F}_S \to G$, $s \mapsto s$ $(s \in S)$

goal : use S, R to construct a space X

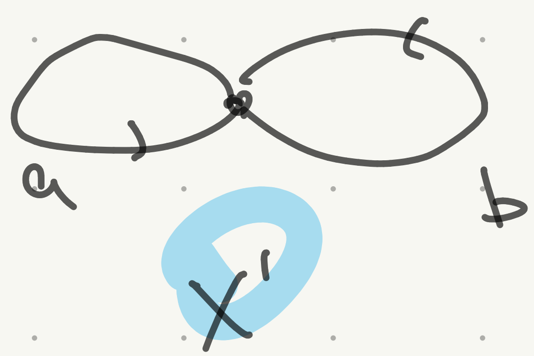
s.t. $\pi_1(X, x_0) \cong G$

realization of X as a 2-dim cell complex

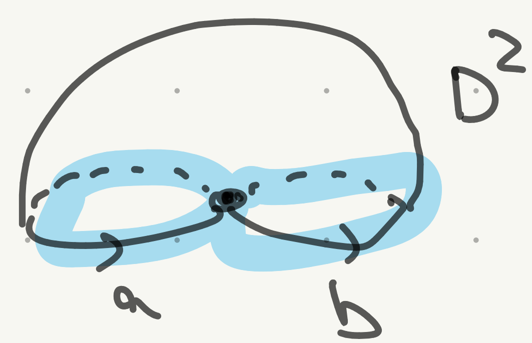
$$X^1 = \underbrace{S^1 \vee \cdots \vee S^1}_{\text{copies labelled by } s \in S}$$

$$X^2 = \left(X^1 \sqcup \left(\coprod_{r \in R} D^2\right)\right) \Big/$$

at r-th component glue $z \in S^1 \subset \partial D^2$ to the cycle in X^1 corresponding to r



$\underbrace{r = ab}$ $\rightsquigarrow$



$\underline{?}$