Section 1.2

Problem 1 structure of $G * H$ $\quad G \neq \{e\} \neq H$

1. $\underline{Z(G*H)} = \{e\}$

center ; $\{ z \in G*H : \forall x \in G*H : zx = xz \}$

$G*H = \{$ reduced words of elements in $G, H \}$

nontriv. elem of $G*H$ is either of the form

- $g_1 h_1 g_2 h_2 \cdots g_k h_k \quad (k \geq 1)$
- $g_1 h_1 g_2 h_2 \cdots g_k h_k g_{k+1} \quad (k \geq 0)$
- $h_1 g_1 h_2 g_2 \cdots h_k g_k \quad (k \geq 1)$
- $h_1 g_1 h_2 g_2 \cdots h_k g_k h_{k+1} \quad (k \geq 0)$

$g_i \in G \setminus \{e\}$

$h_i \in H \setminus \{e\}$

Pattern 1 : $x = g_1 h_1 g_2 h_2 \cdots g_k h_k$ as above

cannot commute with $h \in H \setminus \{e\}$

$h \cdot x = h g_1 h_1 \cdots h_k$ as a reduced word

$x \cdot h$ is represented by a reduced word

starting with g_1

Pattern 2 : $x = g_1 h_1 g_2 h_2 \cdots g_k h_k g_{k+1}$ as above

cannot commute with $h \in H \setminus \{e\}$

$x \cdot h = g_1 h_1 g_2 h_2 \cdots g_k h_k g_{k+1} h$ as reduced word

$h \cdot x = h g_1 h_1 g_2 h_2 \cdots g_k h_k g_{k+1}$ as reduced word

Patterns 3 & 4 : switch the role of G & H

2. $x \in G * H$ has finite order ($x^n = e$ for some $n > 0$)

$\Leftrightarrow$ x is conjugate to some $g \in G$ or $h \in H$ of finite order

$\Leftarrow$ is obvious (yxy^{-1} has same order as x)

$\Rightarrow$ enough to show that: if x is not conjug. to any of $g \in G$ or $h \in H$, then x has infinite order

let x' be a conjugate of x with the minimal length (for the reduced expression)

Part 1 $\quad x' = g_1 h_1 g_2 h_2 \cdots g_k h_k$ or

$x' = h_1 g_1 h_2 g_2 \cdots h_k g_k$ (i.e. x' has even len.)

otherwise; if $x' = g_1 h_1 g_2 h_2 \cdots g_k h_k g_{k+1}$

$x'' = g_{k+1} x' g_{k+1}^{-1}$ is a conjug. of x,

$x'' = (g_{k+1} g_1) h_1 g_2 \cdots h_k$ is shorter than x'

$\Rightarrow$ contradicts with minimality

similarly for the case $x' = h_1 g_1 h_2 g_2 \cdots h_k g_k h_{k+1}$

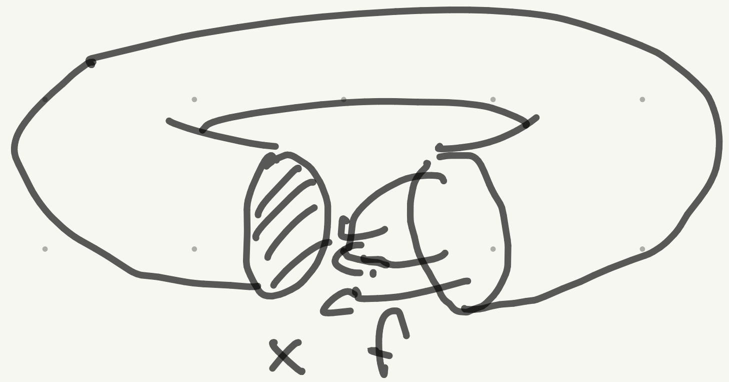
Part 2 $\quad y = g_1 h_1 g_2 h_2 \cdots g_k h_k$ has infinite order

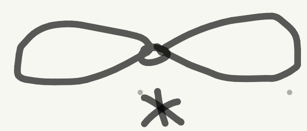
$y^2 = g_1 h_1 g_2 h_2 \cdots g_k h_k g_1 h_1 g_2 h_2 \cdots g_k h_k$ (reduced) (cont.) y^m has similar reduced expression of length $2mk$ $(m \geq 1)$ $\Rightarrow y^m \neq e$

Part 2' $y = h_1 g_1 h_2 g_2 \cdots h_k g_k$ has infinite order.

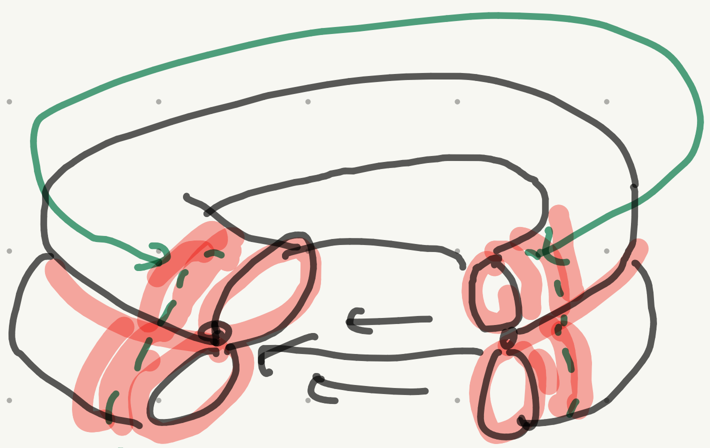
Problem 11 fundamental group of mapping torus

$f : X \to X \rightsquigarrow T_f = X \times I / (x,0) \sim (f(x),1)$



1. $X = S^1 \vee S^1$ 

f basepoint pres.



contr. neigh. of $*$

A: image of $\left(X \times ([0, 0.2) \cup (0.8, 1])\right) \cup (U \times I)$

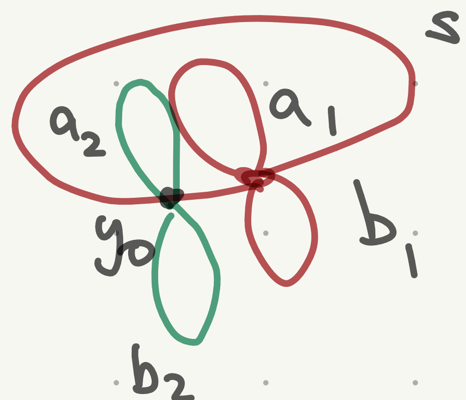
B: image of $X \times (0.1, 0.9)$

$y_0 \in A \cap B$ $\leftarrow$ path-connected

Van Kampen's th'm gives

$$\pi_1(T_f, y_0) \cong \pi_1(A, y_0) \underset{\pi_1(A\cap B, y_0)}{*} \pi_1(B, y_0)$$

A has $S_1 \vee S_1 \vee S_1$ as deform. retract

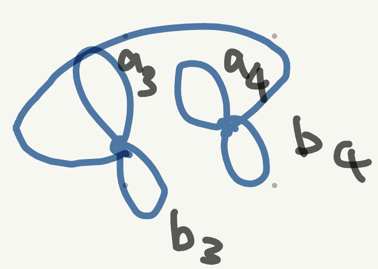


$\rightsquigarrow \pi_1(A, y_0) \cong \mathbb{F}_3$

B has $S^1 \vee S^1$ as deform. retr.

$\rightsquigarrow \pi_1(B, y_0) \cong \mathbb{F}_2$

$A \cap B$ has $S^1 \vee S^1 \vee S^1 \vee S^1$ as def. retr.



$\pi_1(A \cap B, y_0) \to \pi_1(A, y_0)$ is

$a_3 \mapsto a_1$, $b_3 \mapsto b_1$, $a_4 \mapsto s f_*(a_1) s^{-1}$, $b_4 \mapsto s f_*(b_1) s^{-1}$

$\pi_1(A \cap B, y_0) \to \pi_1(B, y_0)$ is

$a_3 \mapsto a_2$, $b_3 \mapsto b_2$, $a_4 \mapsto a_2$, $b_4 \mapsto b_2$

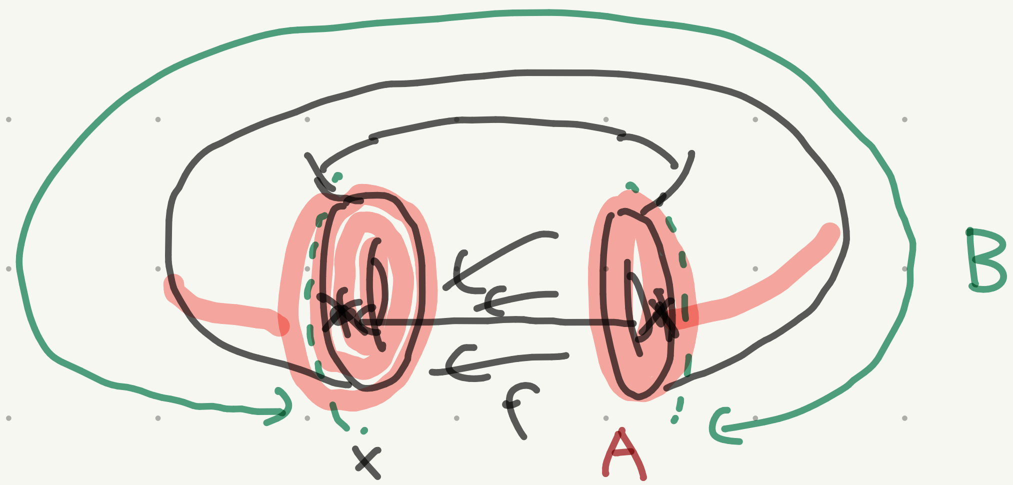
so $\pi_1(T_f, y_0) \cong \langle a_1, b_1, s :$

$$s f_*(a_1) s^{-1} = a_1,\ s f_*(b_1) s^{-1} = b_1 \rangle$$

$$\cong \langle \mathbb{F}_2, \underset{\substack{\shortparallel \\ s^{-1}}}{t} : t x t^{-1} = f_*(x) \quad (x \in \mathbb{F}_2) \rangle$$

the <u>HNN</u> <u>extension</u> of $\mathbb{F}_2 = \pi_1(S^1 \vee S^1, x)$ for f_*

2. $X = S^1 \times S^1$ formally similar



contr. neigh. of $*$

A: image of $\left[X \times ([0, 0.2) \cup (0.8, 1])\right] \cup (U \times I)$

B: image of $X \times (0.1, 0.9)$

$y_0 \in A \cap B$ $\leftarrow$ path-connected

$\rightsquigarrow$ A has $X \vee S^1$

B has X

$A \cap B$ has $(X \cup I \cup X)/$ glue endpt of I to basept of X on both ends

as deform retr.

(cont.) so $A \cap B \simeq X \vee X$ by shrinking I

with $y_0 \in A \cap B$

$$\pi_1(A, y_0) \cong \pi_1(X, y_0) * \underbrace{\mathbb{Z}}_{= \langle s \rangle}$$

$$\pi_1(B, y_0) \cong \pi_1(X, y_0)$$

$$\pi_1(A \cap B, y_0) \cong \underbrace{\pi_1(X, y_0)}_{= \{g^{(1)} : g \in \pi_1(X, y_0)\}} * \underbrace{\pi_1(X, y_0)}_{= \{g^{(2)} : g \in \pi_1(X, y_0)\}}$$

Van Kampen: $\pi_1(T_f, y_0) \cong \pi_1(A, y_0) \underset{\pi_1(A \cap B, y_0)}{*} \pi_1(B, y_0)$

$$\pi_1(A \cap B, y_0) \to \pi_1(A, y_0), \; g^{(1)} \mapsto g, \; g^{(2)} \mapsto s f_*(g) s^{-1}$$

$$\pi_1(A \cap B, y_0) \to \pi_1(B, y_0), \; g^{(1)} \mapsto g, \; g^{(2)} \mapsto g$$

so $$\pi_1(T_f, y_0) \cong \langle \underset{\pi_1(X, y_0)}{\underset{\shortparallel}{\mathbb{Z}^2}}, \underset{s^{-1}}{\underset{\shortparallel}{t}} : t g t^{-1} = f_*(g) \quad (g \in \mathbb{Z}^2) \rangle$$

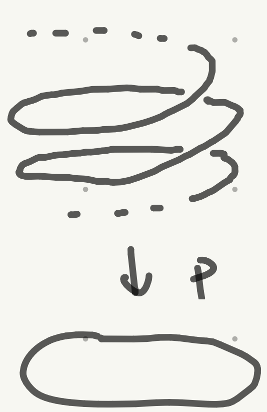
§ 1.3 covering spaces

Recall : for the computation $\pi_1(S^1, *) \cong \mathbb{Z}$

we looked at $\mathbb{R} \xrightarrow{p} S^1$, $t \mapsto (\cos 2\pi t, \sin 2\pi t)$

which was a <u>covering</u>

$$\left(\begin{array}{l} \forall x \in S^1 \ \exists \text{ open } U \ni x \text{ s.t.} \\ p^{-1}(U) = \bigcup_{i \in I} V_i \ ; \ V_i \in \mathbb{R} \text{ open}, \ V_i \cap V_j = \emptyset \ (i \neq j) \\ \qquad\qquad\qquad p|_{V_i} : V_i \to U \text{ homeo} \end{array} \right)$$



$\mathbb{R}$

$\downarrow p$

S^1

and <u>homotopy</u> <u>lifting</u> property

$$\begin{tikzcd} y \arrow[d, mapsto] & Y \arrow[r, "f"] \arrow[d] & \mathbb{R} \arrow[d, shift right] \\ (y,0) & Y \times I \arrow[r, "F"'] \arrow[ur, dashed, "\exists \tilde{F}"] & S^1 \arrow[u, shift right, "p"'] \end{tikzcd}$$

Other coverings of S^1:

$S^1 \to S^1$, $z \mapsto z^m$ $(m = 1, 2, \cdots)$ with $S^1 \subset \mathbb{C}$

corresponding to $\mathbb{R} \xrightarrow{p} S^1$ inducing $\mathbb{R}/m\mathbb{Z} \to S^1$

and $\mathbb{R}/m\mathbb{Z} \cong S^1$

$\rightsquigarrow$ correspondence between

- coverings $X' \xrightarrow{p'} S^1$ s.t. $\begin{tikzcd} \mathbb{R} \arrow[r] \arrow[rr, bend right, "p"'] & X' \arrow[r, "p'"] & S^1 \end{tikzcd}$

- subgroups $m\mathbb{Z} < \mathbb{Z} \cong \pi_1(S^1, *)$

Connection to Galois theory

operations on coverings $\tilde{X} \xrightarrow{p} X$

$\tilde{X}_1 \xrightarrow{p_1} X$, $\tilde{X}_2 \xrightarrow{p_2} X$ two covs.

$\rightsquigarrow$ • disjoint union $\tilde{X}_1 \amalg \tilde{X}_2 \longrightarrow X$

• <u>fiber product</u> $\tilde{X}_1 \times_X \tilde{X}_2 = \{ (x_1, x_2) \in \tilde{X}_1 \times \tilde{X}_2 \mid p_1(x_1) = p_2(x_2) \}$

map of covering spaces : $\tilde{X}_1 \xrightarrow{f} \tilde{X}_2$ s.t. $p_2 \circ f = p_1$

$$\left(\begin{tikzcd} \tilde{X}_1 \arrow[rr, "f"] \arrow[dr] & \arrow[d, phantom, "\circlearrowright"] & \tilde{X}_2 \arrow[dl] \\ & X & \end{tikzcd} \right)$$

$\rightsquigarrow$ <u>monoidal</u> <u>category</u> with fiber product

$x_0 \in X \rightsquigarrow \tilde{X}_{x_0} = p^{-1}(x_0) \subset \tilde{X}$ discrete set

<u>fiber</u> at x_0

$F(\tilde{X}, p) = \tilde{X}_{x_0}$ is a <u>functor</u> on the category of covering spaces over X

s.t. $F(\tilde{X}_1 \times_X \tilde{X}_2 ; p_1 \times p_2) \overset{(\sim)}{=} F(\tilde{X}_1, p_1) \times F(\tilde{X}_2, p_2)$

i.e. (strict) <u>monoidal</u> functor

$$\begin{pmatrix} \text{(symmetric) mon. cat. of} \\ \text{objs}: \tilde{X} \xrightarrow{p} X \\ \text{mors}: \begin{tikzcd} \tilde{X}_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & \tilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd} \\ \text{mon. prod}: \tilde{X}_1 \times_X \tilde{X}_2 \end{pmatrix} \xrightarrow{F} \begin{pmatrix} \text{monoidal cat. of} \\ \text{obsj}: \text{sets } Y \\ \text{mors}: \text{maps } Y_1 \to Y_2 \\ \text{mon. prod}: Y_1 \times Y_2 \end{pmatrix}$$

Grothendieck's observation: this structure is "same" as

$$\begin{pmatrix} \text{(symmetric) monoidal cat. of} \\ \text{objs}: \ G \curvearrowright Y \ \ \text{action on sets} \\ \text{mors}: \ Y_1 \to Y_2 \ \ \text{comm. with } G\text{-actions} \\ \text{mon. prod.}: \ Y_1 \times Y_2 \ \ \text{diag. action} \end{pmatrix} \xrightarrow{F} \left(\text{mon. cat. of sets}\right)$$

$$\ni (G \curvearrowright Y) \mapsto \underset{\psi}{Y}$$

"forget the action of G"

for $G = \pi_1(X, x_0)$

We can recover G as $\mathrm{Aut}^{\otimes}(F) = \left\{ \eta_{(\tilde{x},p)} : (\tilde{x},p) \to (\tilde{x},p) \ \ F(\eta_{\tilde{x}_1 \times \tilde{x}_2}) = F(\eta_{\tilde{x}_1}) \times F(\eta_{\tilde{x}_2}) \right\}$

... which has analogy to

$$\left(\begin{array}{ll} \text{symm. mon. category of} & \\ \text{objs : separable algs } A/K & \\ \text{mors : } A \to B & \text{hom of} \\ & K\text{-algs} \\ \text{mon. prod : } A \otimes_K B & \end{array}\right) \xrightarrow{F} (\text{mon. cat. of sets})$$

$$\begin{array}{c} \Psi \\ \ni A \mapsto \mathrm{Hom}_K(A, \bar{K}) \end{array}$$

K : field , $\bar{K}/K$: alg. closure

$\mathrm{Gal}(\bar{K}/K) = \mathrm{Aut}_K(\bar{K})$ (absolute) Galois group

acting on $F(A) = \mathrm{Hom}_K(A, \bar{K})$