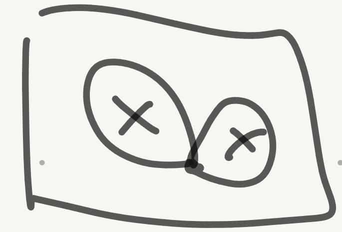
Section 1.2 Exercises



Problem 3 $F \subset \mathbb{R}^n$ finite subset

$\Rightarrow \mathbb{R}^n \setminus F$ is simply connected if $n \geq 3$

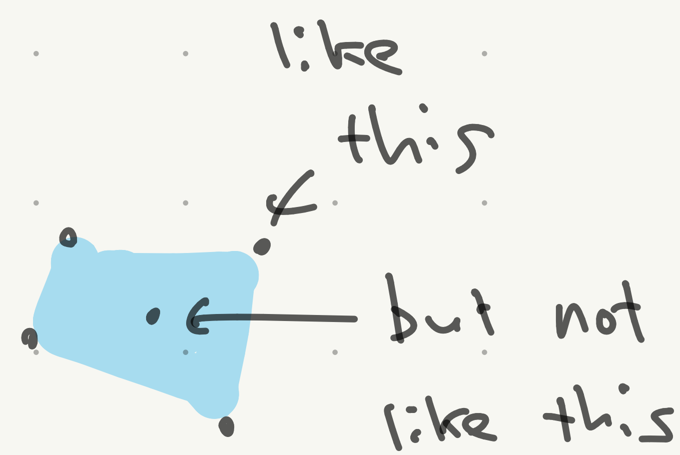
Proof by induction on $k = |F|$

$k=0$: $\mathbb{R}^n$ is contractible

$k=1$: $\mathbb{R}^n \setminus \{pt\}$ is homotopy equivalent to S^{n-1}

S^m is simply conn. for $m \geq 2$

general case :



Step 1. remove a point from F

take a convex hull of F :

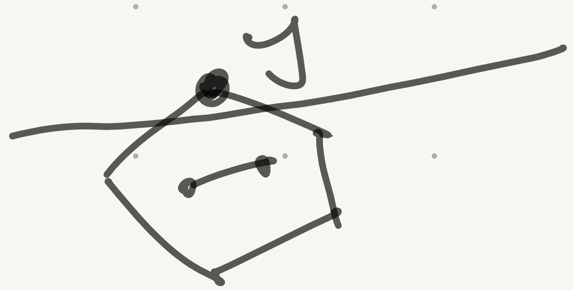
$$C = \{ \sum_{x \in F} t_x \cdot x \;:\; t_x \geq 0, \sum_{x \in F} t_x = 1 \} \subset \mathbb{R}^n$$

(cont.) C closed convex in $\mathbb{R}^n$

$\Rightarrow$ C is the convex hull of its <u>extremal</u> <u>points</u>

($y \in C$ extremal $\overset{\text{def}}{\equiv}$ $z, w \in C$, $0 \leq t \leq 1$: $y = tz + (1-t)w$

$\Rightarrow$ $z = w = y$)



extremal points of C $\subset F$

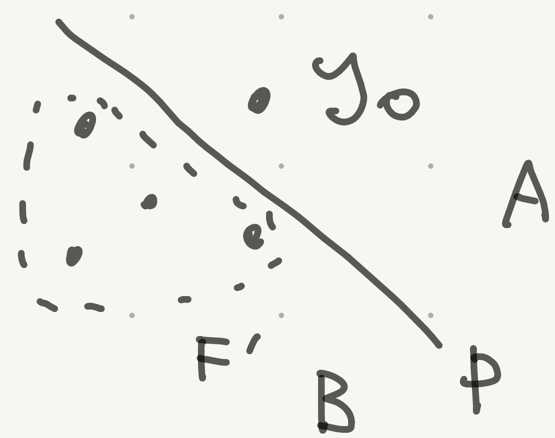
$\rightsquigarrow$ take $y_0 \in F$ which is extremal in C

$\Rightarrow$ $\exists$ hyperplane $P \subset \mathbb{R}^n$ separating y_0 from the convex hull of $F' = F \setminus \{y_0\}$

Step 2 reduction by gluing

P splits $\mathbb{R}^n$ to two half-spaces A, B

$y_0 \in A$, $F' \subset B$, $A \simeq \mathbb{R}^n \simeq B$



with $x_0 \in P$, Van Kampen's th'm says

$$\pi_1(\mathbb{R}^n \setminus F, x_0) \cong \pi_1(\overline{A \setminus \{y_0\}}, x_0) \underset{\pi_1(P, x_0)}{*} \pi_1(\overline{B \setminus F'}, x_0)$$

$\pi_1(\overline{A \setminus \{y_0\}}, x_0) \cong \pi_1(S^{n-1}, *)$ trivial

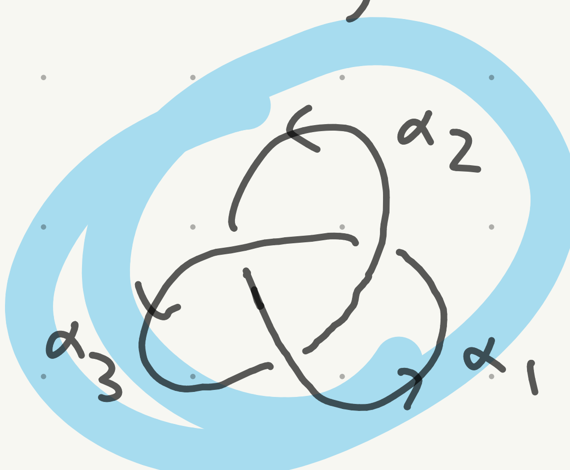
$\pi_1(\overline{B \setminus F'}, x_0) \cong \pi_1(\mathbb{R}^n \setminus F', x_0)$ trivial by induction hypo.

Problem 22 presentation of $\pi_1(\mathbb{R}^3 \setminus K, *)$ (knot group)

for (oriented) knots $K \subset \mathbb{R}^3$

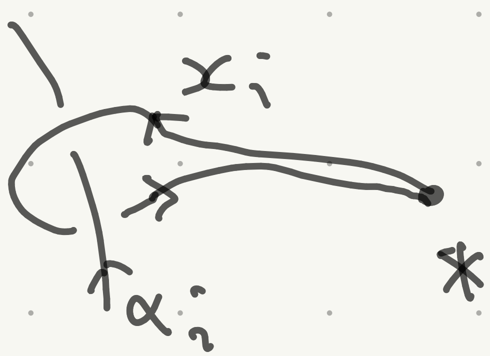
basic idea

take "regular projection" onto a plane



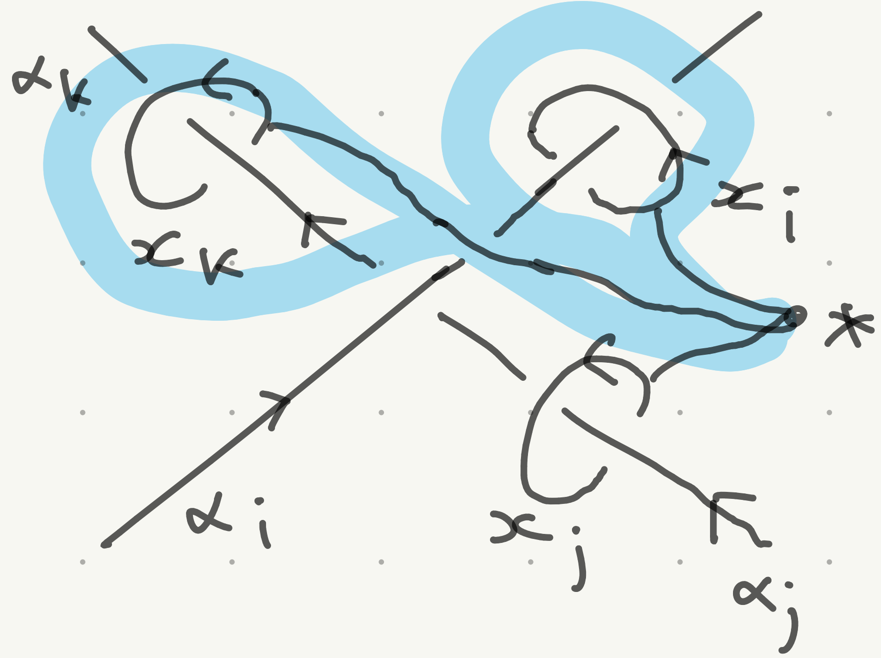
and label segments $\alpha_1, \alpha_2, \ldots$

each segment gives an element of $\pi_1(\mathbb{R}^3 \setminus K, *)$



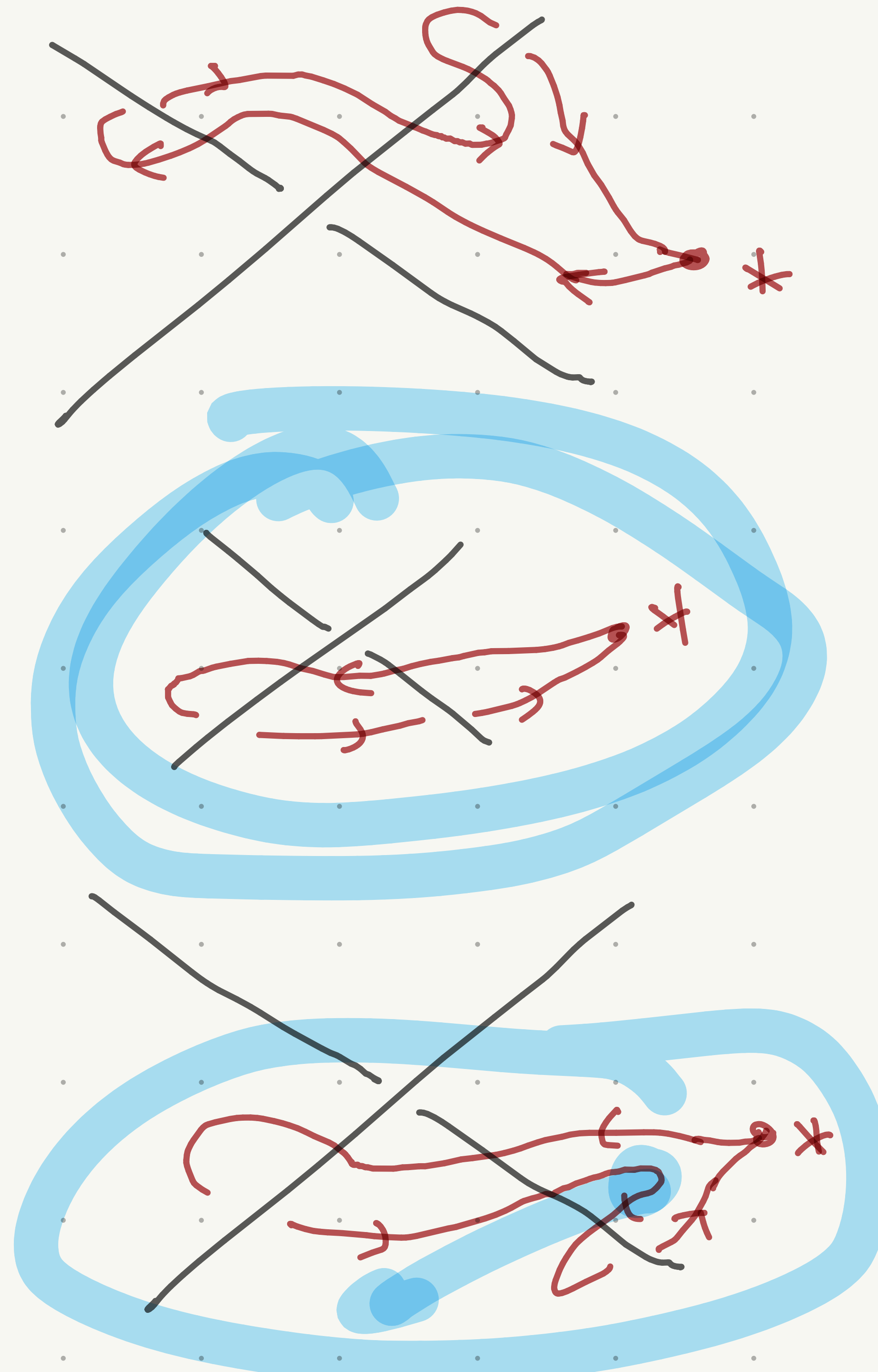
x_i : go around α_i "counterclockwise" when we look in the direction of α_i

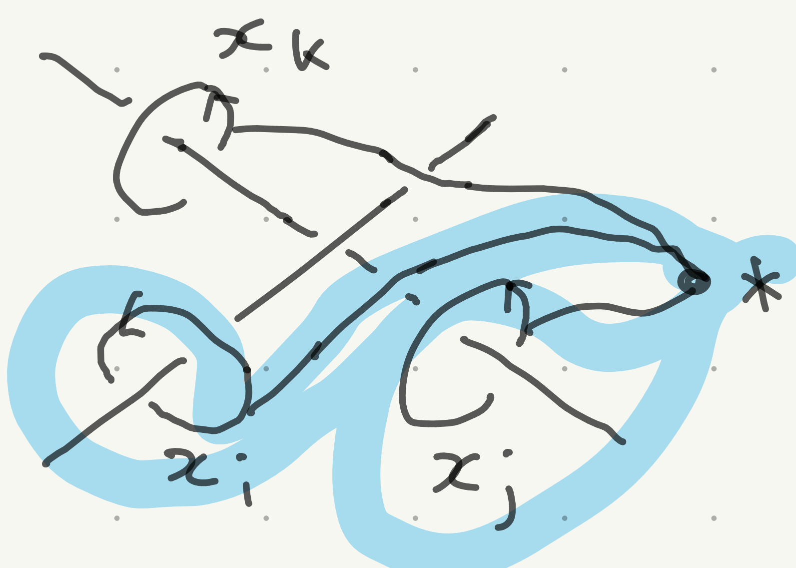
⇝ at each crossing of segments we get a relation



$x_k \cdot x_i$

⇝





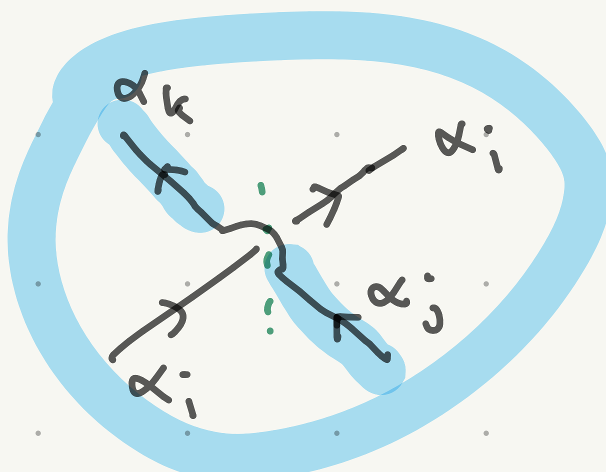
$x_i \cdot x_j$

⇝ <u>Wirtinger</u> <u>presentation</u>: "$x_i x_k = x_j x_i$" at each crossing

More formally: construct a 2-dim cell complex X

s.t. $X \simeq \mathbb{R}^3 \setminus K$

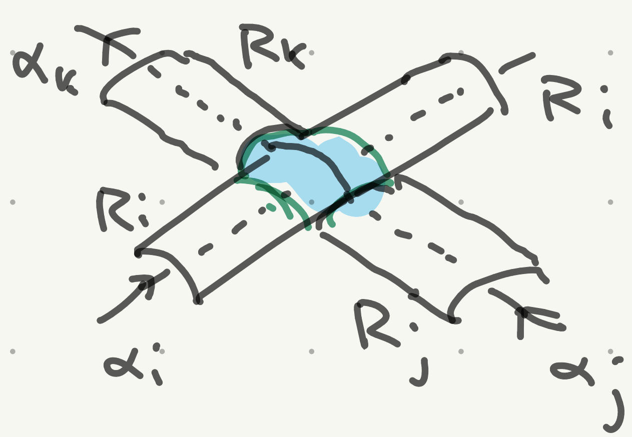
: change labeling convention of segments



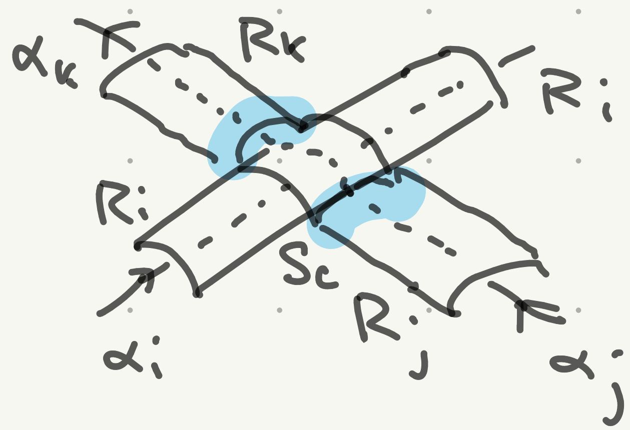
(cut the "overcrossing" ones)

put rectangle "cover" R_i on each segment α_i,

still exposing the crossings



then put a rectangle cover S_c



on each crossing c

.. and close the base of R_i with another rectangle R_i'

$$X = \Big(\coprod_{i:\text{ label for segments}} (R_i \amalg R_i') \Big) \amalg \Big(\coprod_{c:\text{ label for crossings}} S_c \Big) \Big/ \text{gluing}$$

$T_i = (R_i \amalg R_i' / \text{gluing}) \cong S^1 \times I$

gives $x_i \in \pi_1(T_i, *)$

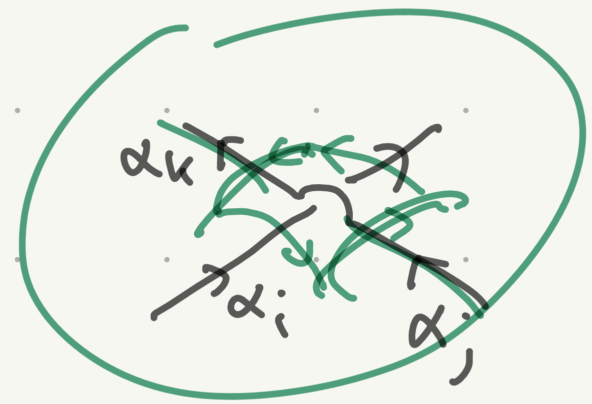
attaching $S_c \cong D^2$ to $\bigcup_i T_i$ introduces

a relation

$x_i x_k x_i^{-1} x_j^{-1} = e$

boundary of S_c

when c is



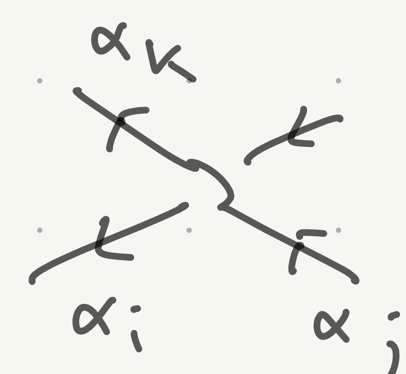
(Feb 17

$\pi_1(X \cup D^2, *)$

$\cong \pi_1(X, *) / \langle$ boundary loop of D^2)

$x_i^{-1} x_k x_i x_j^{-1} = e$

when c is



- $\pi_1(\mathbb{R}^3 \setminus K, *)^{ab} \cong \mathbb{Z}$ $\quad$ ($G^{ab} = G/[G,G]$)

abelianization

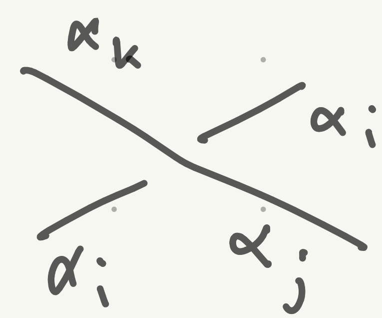
$$x_i x_k = x_j x_i \iff x_i x_k x_i^{-1} = x_j$$

in the abelianization ($\bar{x}$: img of x)

$\overline{x_i}\,\overline{x_k} = \overline{x_k}\,\overline{x_i}$ so we get

$$\bar{x}_k \bar{x}_i = \bar{x}_j \bar{x}_i \Rightarrow \overline{x_k} = \overline{x_j}$$

from each crossing



K is connected $\Rightarrow$ $\overline{x_k} = \overline{x_j}$ for all j, k

"winding number around K" still remembers $\mathbb{Z}$

Lifting properties for covering

Recall (Feb. 2, "Proposition c)")

If $p : \tilde{X} \to X$ is a covering,

$$\begin{tikzcd} y \arrow[d, maps to] & Y \arrow[r, "f"] \arrow[d] & \tilde{X} \arrow[d, "p"] \\ (y,0) & Y \times I \arrow[r, "F"'] \arrow[ur, dashed, "\exists ! \tilde{F}"] & X \end{tikzcd}$$

for any f & F making this diagram comm., there is unique $\tilde{F}$ making it comm.

(Homotopy Lifting Property ; fibration)

Prop (1.31) $p : \tilde{X} \to X$ covering, $\tilde{x}_0 \in \tilde{X}$, $x_0 = p(\tilde{x}_0)$

then the induced hom. $\pi_1(\tilde{X}, \tilde{x}_0) \xrightarrow{p_*} \pi_1(X, x_0)$

is injective

Proof Take $\tilde{f}_0 : (S^1, *) \to (\tilde{X}, \tilde{x}_0)$ (loop at $\tilde{x}_0$)

s.t. $p_*([\tilde{f}_0]) = e$ cont. map $S^1 \to \tilde{X}$ s.t. $\tilde{f}_0(*) = \tilde{x}_0$

This means there is a homotopy of paths

$F : I \times I \to X$ between

$p \circ \tilde{f}_0 : I \to S^1 \to X$ and the const. path at x_0

$$\begin{tikzcd} I \arrow[r, "\tilde{f}_0"] \arrow[d] & \tilde{X} \arrow[d] \\ I \times I \arrow[r, "F"'] \arrow[ur, dashed, "\tilde{F}?"] & X \end{tikzcd}$$

$\tilde{F}$ is a homotopy of paths from $\tilde{f}_0$ to const. path $\tilde{x}_0$

$\tilde{F}(s,t) \in \tilde{X}$ cont in s,t

$\tilde{F}(0,t) \in \underbrace{p^{-1}(x_0)}_{\text{discrete subset of } \tilde{X}}$ from $F(0,t) = x_0$

($\tilde{X} \xrightarrow{p} X$ covering $\Rightarrow$ $\exists\, U \ni x_0$ s.t.

$p^{-1}(U) = \coprod_{i \in I} V_i$ $\quad V_i$ open, mutually disjoint)

$\Rightarrow$ $\tilde{F}(0,t)$ stays at $\tilde{F}(0,0) = \tilde{x}_0$

$\uparrow$

$\tilde{f}_0(0)$

similarly $\tilde{F}(s,1) \in p^{-1}(x_0) \Rightarrow \tilde{F}(s,1) = \tilde{x}_0$

Prop (1.32) $p : \tilde{X} \to X$ covering, $\tilde{x}_0 \overset{p}{\mapsto} x_0$

suppose $\tilde{X}$ is path-connected ($\Rightarrow$ so is X)

then $\underbrace{[\pi_1(X, x_0) : \pi_1(\tilde{X}, \tilde{x}_0)]}_{\text{index of the subgroup}} = \underbrace{|p^{-1}(x_0)|}_{\text{covering degree}}$

$\pi_1(\tilde{X}, \tilde{x}_0) < \pi_1(X, x_0)$

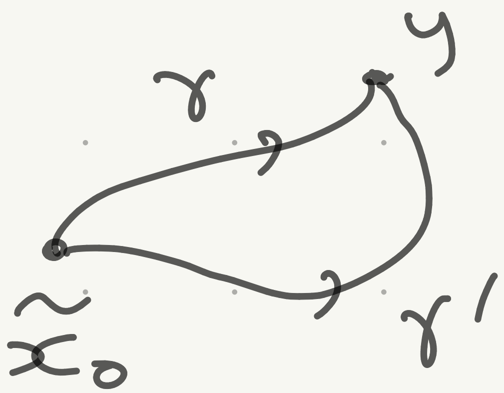
Proof

$y \in p^{-1}(x_0) \Rightarrow \exists$ path γ from $\tilde{x}_0$ to y

(by path-conn.ess of $\tilde{X}$)

γ, γ' : paths from $\tilde{x}_0$ to y

$\Rightarrow \gamma' \simeq h \cdot \gamma$ for some loop at $\tilde{x}_0$



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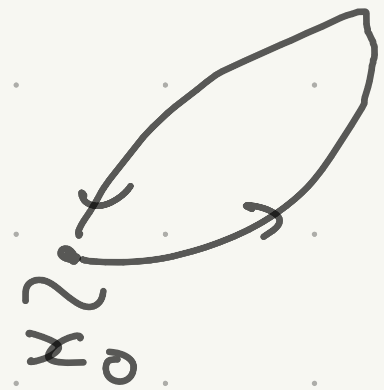
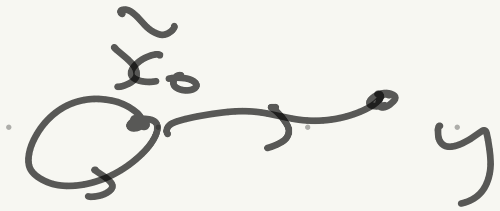


gives $h \cdot \gamma \simeq \gamma'$

So

$S = \{[\gamma] : \gamma : \text{path from } \tilde{x}_0 \text{ to some } y \in p^{-1}(x_0)\}$

homotopy class of paths



has action of $H = \pi_1(\tilde{X}, \tilde{x}_0)$, $|H \backslash S| = |p^{-1}(x_0)|$

Any loop $g : (S^1, *) \to (X, x_0)$ lifts to a path γ from $\tilde{x}_0$ to some $y \in p^{-1}(x_0)$

$G = \pi_1(X, x_0) \to S$, $[g] \mapsto [\gamma]$

lift of g

is bijective by uniqueness of lift, homotopy..

surjectivity: take any $[h] \in \pi_1(\tilde{X}, \tilde{x}_0)$

get $g = p \circ h$ loop at x_0

lift to a path h' starting at $\tilde{x}_0$

h and h' both lift g, with same starting pt

$\Rightarrow h = h'$

this gives identification of sets with

H-action $\Rightarrow |H \backslash G| = |H \backslash S|$

$[G:H]$ $|p^{-1}(x_0)|$

"Unique lifting property"

Prop (1.34) $p : \tilde{X} \to X$ covering, Y connected

$f : Y \to X$ map, $\tilde{f}_1, \tilde{f}_2 : Y \to \tilde{X}$ lifts of f

$\exists\, y \in Y$ $\tilde{f}_1(y) = \tilde{f}_2(y) \Rightarrow \tilde{f}_1 = \tilde{f}_2$

Proof $A = \{y' : \tilde{f}_1(y') = \tilde{f}_2(y')\}$ is closed, nonempty

$y \in A$

it's enough to show A is open (by connectedness of Y)

$y' \in A$ $\rightsquigarrow$ take evenly covered open neigh.

U of $p(\tilde{f}_1(y')) = p(\tilde{f}_2(y'))$

So $\quad p^{-1}(U) = \coprod_{i \in I} V_i \quad V_i$ open in $\hat{X}$,

$V_i \cap V_j = \emptyset$,

$p|_{V_i} : V_i \to U$ homeo

$\tilde{f}_1(y') = \tilde{f}_2(y') \in V_{i_0}$ for some i_0

V_{i_0} is open $\Rightarrow$ $\tilde{f}_1^{-1}(V_{i_0})$ is an open neigh. of y'

similarly with $\tilde{f}_2^{-1}(V_{i_0})$

$\Rightarrow$ $W = \tilde{f}_1^{-1}(V_{i_0}) \cap \tilde{f}_2^{-1}(V_{i_0})$ is also

$\tilde{f}_1(y'') = \tilde{f}_2(y'')$ for $y \in W$ since both map to $f(y'')$ under p, $p|_W$ is injective