Classification of covering spaces

Overview

X space (with some "regularity"), $x_0 \in X$

$\rightsquigarrow$ bijective correspondence

$$\left(\begin{array}{l} \text{coverings over } X \\ p : Y \to X \\ \forall x \in X \ \exists \underset{\text{open}}{U} \ni x \\ p^{-1}(U) = \coprod_{i \in I} V_i \end{array} \right) \overset{1:1}{\longleftrightarrow} \left(\begin{array}{l} \text{subgroups of } \pi_1(X, x_0) \\ H < \pi_1(X, x_0) \end{array} \right)$$

$$(Y, p) \rightsquigarrow \pi_1(Y, \tilde{x}_0) \ \left(\xhookrightarrow{\text{induced by } p} \pi_1(X, x_0) \right)$$

$p(\tilde{x}_0) = x_0$

In general (Prop 1.31)

$Y \xrightarrow{p} X$ covering, $p(\tilde{x}_0) = x_0$

$\rightsquigarrow$ $p_*: \pi_1(Y, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective

So we need to figure out a reasonable assumption on X s.t. the inverse correspondence holds

- X should be (path) connected

  ($\pi_1(X, x_0)$ does not "see" other components not containing x_0)

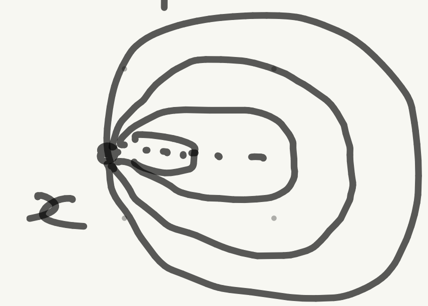
- there should be a distinguishd covering $\tilde{X} \xrightarrow{p} X$ corresp. to $\{e\} < \pi_1(X, x_0)$; i.e. $\tilde{X}$ simply connected

We want X to be locally path-connected and

- locally contractible: $\forall x \in X$, $\underset{\text{open}}{V} \ni x$ $\exists V \supset \underset{\text{open}}{U} \ni x$

  U is contractible

not



(cell complexes are locally contractible)

or, a bit more generally

- locally simply connected: $\forall x \in X$, $\underset{\text{open}}{V} \ni x$ $\exists V \supset \underset{\text{open}}{U} \ni x$

  $\pi_1(U, x) = \{e\}$

or, even yet more generally

- semilocally simply connected: $\forall x$, $V \ni x$ $\exists V \supset U \ni x$

  $\pi_1(U, x) \to \pi_1(X, x)$ is trivial

Universal covering

Def A <u>universal covering</u> of X is a covering $p : \tilde{X} \to X$ s.t. $\pi_1(\tilde{X}, \tilde{x}_0)$ is trivial (for all $\tilde{x}_0$)

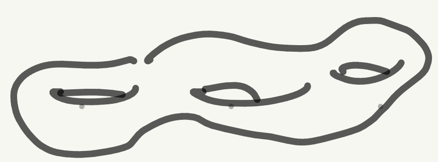
Examples

1. $X = S^1 \rightsquigarrow \mathbb{R} \xrightarrow{p} S^1$ is universal

2. X : closed oriented surface
(without boundary)

$X = S^2 \rightsquigarrow \tilde{X} = S^2 \quad p = id$

$X = M_g \ (g \geq 1) \rightsquigarrow \tilde{X} = \mathbb{H} \cong \mathbb{R}^2$



"uniformization of Riemann surfaces"

Construction of universal covering

X path-connected and locally contractible

(or locally path-conn., semilocally simply connected)

$x_0 \in X$

$\tilde{X} = \{ \underbrace{[\gamma]}_{} : \text{path in } X, \gamma(0) = x_0 \} = \pi_1(X)^{x_0}$

equiv. class for homotopy of paths

(basept-fixing cont. deform.)

- $p : \tilde{X} \to X$ is given by $[\gamma] \mapsto \gamma(1)$

- roughly : "$H([\gamma], t) = \gamma(t)$" gives cont.
  deform. between const. map and $\mathrm{id}_{\tilde{X}}$
  (but this is too rough, we only get $\pi_1(\tilde{X}, \text{const}) = 0$)

Topology on $\tilde{X}$

Want : $p : \tilde{X} \to X$, $[\gamma] \mapsto \gamma(1)$ is a local homeomorphism ; i.e.

- given $x \in X$ we should give $\underset{\text{open}}{U} \ni x$ (small)
- and write $p^{-1}(U) = \coprod_{i \in I} V_i$ for some coll. of subsets $V_i \subset \tilde{X}$
- then say these V_i are open sets of $\tilde{X}$
- and check consistency as base of topology (for all V_i's from different U, x)

As candidates of U:

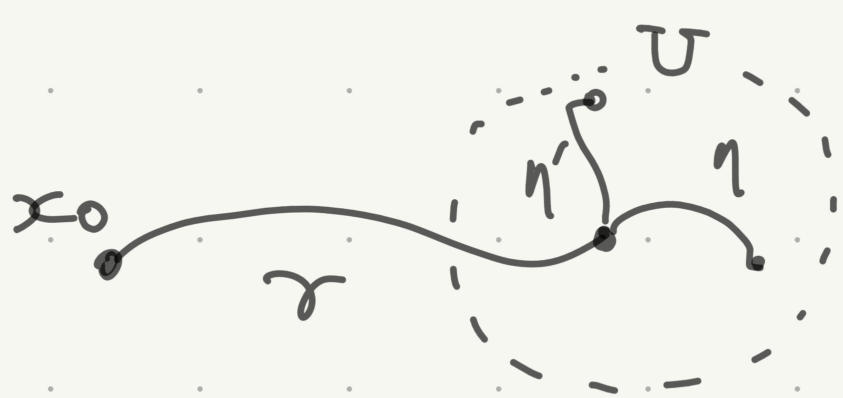
$\mathcal{U} = \{ U \subset X$ contractible & open $\}$

(or open, path-conn.,
$\pi_1(U, x) \to \pi_1(X, x)$ triv. for any $x \in U$)

$\rightsquigarrow \forall x \in X \ \exists U \in \mathcal{U}$ s.t. $x \in U$

Fix $[\gamma] \in \tilde{X}$; for each $U \in \mathcal{U}$ s.t. $\gamma(1) \in U$

put $U_{[\gamma]} = \{ [\gamma \cdot \eta] : \eta$ path in U, $\gamma(1) = \eta(0) \}$



Lem. 1 $\quad p|_{\bar{U}_{[\gamma]}} : \bar{U}_{[\gamma]} \to U$ is bijective

- surj : $\forall y \in U$ can be connected to $\gamma(1)$ by a path in U
- inj : $p([\gamma \cdot \eta]) = p([\gamma \cdot \eta'])$ means $\eta(1) = \eta'(1)$

$\rightsquigarrow [\eta] = [\eta']$ ($\exists$ homotopy of paths) by assumption on U

Lem. 2 $\quad [\gamma'] \in \bar{U}_{[\gamma]} \Rightarrow \bar{U}_{[\gamma]} = \bar{U}_{[\gamma']}$

$[\gamma'] = [\gamma \cdot \eta] \Rightarrow [\gamma] = [\gamma' \cdot \underbrace{\bar{\eta}}_{\text{flip}}]$

Prop. 1. $\tilde{\mathcal{U}} = \{ U_{[\gamma]} : U \in \mathcal{U}, [\gamma] \in \tilde{X}, \gamma(1) \in U \}$

form a base of topology on $\tilde{X}$

i.e. $U_{[\gamma]}, V_{[\gamma']} \in \mathcal{U} \Rightarrow \exists\, W_{[\gamma'']} \subset U_{[\gamma]} \cap V_{[\gamma']}$

Proof Take $[\gamma''] \in U_{[\gamma]} \cap V_{[\gamma']}$

Then $U_{[\gamma]} = U_{[\gamma'']}$, $V_{[\gamma']} = V_{[\gamma'']}$

$W = U \cap V$ satisfies $W_{[\gamma'']} \subset U_{[\gamma'']} \cap V_{[\gamma'']}$

$\rightsquigarrow$ We get topology on $\tilde{X}$, $p : \tilde{X} \to X$ cont.

Prop. 2. $p : \tilde{X} \to X$ is a covering

Proof Fix $x \in X$ take $U \in \mathcal{U}$ s.t. $x \in U$

We get sets $U_{[\gamma]}$ for $[\gamma] \in \tilde{X}$, $\gamma(1) \in U$

There is (huge) redundancy, but

$[\gamma] \equiv [\gamma'] : \quad U_{[\gamma]} = U_{[\gamma']} \Leftrightarrow [\gamma'] \in U_{[\gamma]}$

is an equiv. relation for such γ's

$\rightsquigarrow$ take a set of representatives as I

$$p^{-1}(U) = \underbrace{\coprod_{[\gamma] \in I} U_{[\gamma]}}$$ mutually disjoint

Prop. 3 $\tilde{X}$ is path-connected

Fix $[\gamma] \in X$ then $\Gamma : I \to \tilde{X}$, $\Gamma(t) = [s \mapsto \gamma(ts)]$

is continuous, represents a path connecting

$[\gamma]$ to [const. at x_0] in $\tilde{X}$

Prop. 4 $\pi_1(\tilde{X}, \tilde{x}_0) = \{e\}$ for $\tilde{x}_0 =$ [const. at x_0]

Take a loop Γ in $\tilde{X}$ based at $\tilde{x}_0$, $\Gamma(s) = [\gamma_s]$

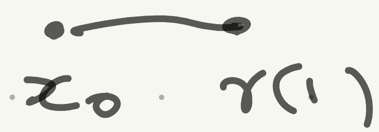
$\rightsquigarrow \gamma'(s) = \gamma_s(1)$ is a loop based at x_0

- $[\gamma'] = [x_0]$ by the loops $\gamma'_t(s) = \gamma_s(t)$

- $p_*([\Gamma]) = [\gamma']$, p_* is injective

So far :

$$\tilde{X} = \{[\gamma] : \gamma(0) = x_0\} = \{ [\gamma] \in \Pi_1(X),\ r([\gamma]) = x_0 \}$$



Note $\pi_1(X, x_0)$ acts on $\tilde{X}$ by product

$$\underset{\pi_1(X,x_0)}{\underset{\cap}{[\gamma]}} \cdot \underset{\tilde{X}}{\underset{\cap}{[\gamma']}} = [\gamma \cdot \gamma'] \in \tilde{X}$$

$\rightsquigarrow$ given $H < \pi_1(X, x_0)$ we want to take $X_H := H \backslash \tilde{X}$

and check :

- $X_H \to X$ covering (induced by $\tilde{X} \xrightarrow{p} X$)
- $\pi_1(X_H, [\tilde{x}_0]) \cong H$

Recall $p^{-1}(U) = \coprod_{[\gamma'] \in I} U_{[\gamma']}$ for $U \in \mathcal{U}$ open contr.

the action $\pi_1(X, x_0) \curvearrowright \tilde{X}$ preserves $p^{-1}(U)$

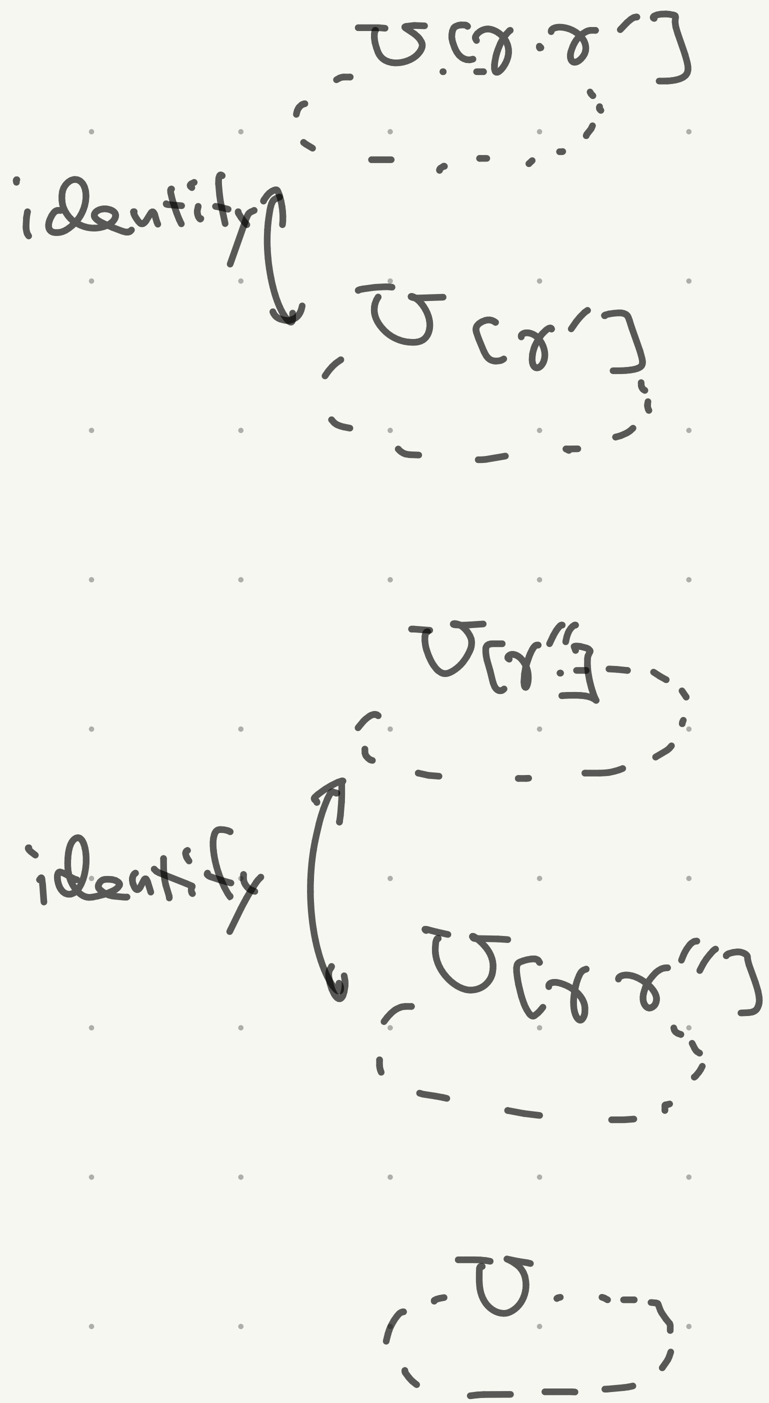
and permutes the components $U_{[\gamma']}$

(<u>deck transform</u>)

$[\gamma] \cdot U_{[\gamma']} = U_{[\gamma'']}$ for $[\gamma''] \in I$,

$[\gamma''] \in U_{[\gamma \cdot \gamma']} = [\gamma] \cdot U_{[\gamma']}$

Prop. 4. quotient set $X_H = H \backslash \tilde{X}$, with quotient topology, is a covering over X



$[\gamma] \in H$

Prop 5 $\pi_1(X_H, \underbrace{[\tilde{x}_0]}_{\text{img. of const. path}}) \cong H$ by p_*

Proof Want : $p_*(\pi_1(X_H, [\tilde{x}_0])) = H$

Take $[\gamma] \in \pi_1(X, x_0)$ and look at its lift

$\tilde{\gamma} : I \to \tilde{X}$, $\tilde{\gamma}(0) = \tilde{x}_0$ (unique)

$[\tilde{\gamma}] = [\gamma] \cdot [\tilde{x}_0]$ in $\tilde{X}$

$\rightsquigarrow \hat{\gamma}$: image of $\tilde{\gamma}$ in X_H (path s.t. $\hat{\gamma}(0) = [\tilde{x}_0]$)

is a loop at $[\tilde{x}_0]$ iff $[\gamma] \in H$

On the other hand $p_*([\hat{\gamma}]) = [\gamma]$

We want to say that any covering $Y \xrightarrow{p} X$

is determined by $p_*(\pi_1(Y, \tilde{x}_0)) < \pi_1(X, x_0)$

Prop (1.33 ; "lifting criterion")

$p : \tilde{X}' \to X$ covering , $p(\tilde{x}_0) = x_0$

$f : Y \to X$ cont. $q(y_0) = x_0$

Y : path-conn. & loc. path-conn.

Then $\exists \tilde{f} : Y \to \tilde{X}'$ lift of f , $\tilde{f}(y_0) = \tilde{x}_0$

$\Longleftrightarrow f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{X}', \tilde{x}_0))$

($\Rightarrow$ is obivious from $f = p \circ \tilde{f}$)

$\Leftarrow$:

Step 1 construction of $\tilde{f} : Y \to \tilde{X}'$ as a map

given $y \in Y$ put $\tilde{f}(y) = \widetilde{f \circ \gamma}(1)$ where

- γ : path in Y, from y_0 to y
- $\widetilde{f \circ \gamma}$: lift of path in X, $\widetilde{f \circ \gamma}(0) = \tilde{x}_0$

independence on γ : take another γ'

$$(f \circ \gamma') \cdot \overline{(f \circ \gamma)} = f \circ (\gamma' \cdot \bar{\gamma}) \text{ is a loop at } x_0$$

$\operatorname{img}(f_*) \subset \operatorname{img}(p_*) \Rightarrow \exists [\tilde{h}] \in \pi_1(\hat{X}, \hat{x}_0)$ s.t.

$$[p \circ \tilde{h}] = [(f \circ \gamma') \cdot \overline{(f \circ \gamma)}]$$

$$\begin{tikzcd} s \arrow[d, mapsto] & I \arrow[r, "\tilde h"] \arrow[d] & \tilde X \arrow[d] \\ (s,0) & I\times I \arrow[r, "F"'] \arrow[ur, dashed, "\tilde F"] & X \end{tikzcd}$$

$\rightsquigarrow$ lift of $(f\circ\gamma')\cdot\overline{(f\circ\gamma)}$ in $\tilde X$

as $\tilde F(\sim,1)=\tilde h_1$

F homotop. between $p\circ\tilde h$ and $(f\circ\gamma')\cdot\overline{(f\circ\gamma)}$

$$\tilde h = \widetilde{(f\circ\gamma')}\cdot\overline{\widetilde{(f\circ\gamma)}} \quad \text{so} \quad \widetilde{f\circ\gamma'}(1)=\widetilde{f\circ\gamma}(1)$$

Step 2. $\tilde f$ is continuous

Fix $y\in Y$. take an evenly covered open set

$U\ni f(y)$ $\qquad p^{-1}(U)=\coprod_{i\in I} V_i$

$\exists!\ i_0\in I$ s.t. $\tilde f(y)\in V_{i_0}$ $\qquad p|_{V_{i_0}}: V_{i_0}\xrightarrow[\text{homeo}]{\sim} U$

and $\tilde f=(p|_{V_{i_0}})^{-1}\circ f$ around y

Cor. (Prop 1.37)

X : path-conn. & locally contractible

(or loc. p-c., semiloc. simp. conn.)

$x_0 \in X$

$p_1 : \widetilde{X}_1 \to X$, $p_2 : \widetilde{X}_2 \to X$ coverings

$p_1(\tilde{x}_1) = x_0 = p_2(\tilde{x}_2)$

Then $\exists$ homeo. $f : \widetilde{X}_1 \to \widetilde{X}_2$ s.t. $p_2 \circ f = p_1$, $f(\tilde{x}_1) = \tilde{x}_2$

$\Leftrightarrow$ $(p_1)_*(\pi_1(\widetilde{X}_1, \tilde{x}_1)) = (p_2)_*(\pi_1(\widetilde{X}_2, \tilde{x}_2))$ in $\pi_1(X, x_0)$

Again $\Rightarrow$ is obvious

$\Leftarrow$: lifting criterion gives

$\tilde{p}_1 : \tilde{X}_1 \to \tilde{X}_2$ lift of p_1 $(p_2 \circ \tilde{p}_1 = p_1)$

$\tilde{p}_2 : \tilde{X}_2 \to \tilde{X}_1$ lift of p_2

unique lifting property (Prop. 1.34, Feb 24)

for $\left(\begin{array}{lll} \tilde{p}_2 \circ \tilde{p}_1 \text{ and } id_{\tilde{X}_1} & \Rightarrow & \tilde{p}_2 \circ \tilde{p}_1 = id_{\tilde{X}_1} \\ \tilde{p}_1 \circ \tilde{p}_2 \text{ and } id_{\tilde{X}_2} & \Rightarrow & \tilde{p}_1 \circ \tilde{p}_2 = id_{\tilde{X}_2} \end{array} \right.$

i.e., we can take $f = \tilde{p}_1$