Recall: if $X$ is path-conn. & loc. contractible, $x_0 \in X$

$\exists$ bijective correspondence between

- coverings $p: \underset{\uparrow \text{path-conn.}}{\tilde{X}'} \to X$, basept $\tilde{x}_0' \in p^{-1}(x_0)$
- subgroups $H < \pi_1(X, x_0)$.

concretely:

$$(\tilde{X}' \xrightarrow{p} X, \tilde{x}_0') \rightsquigarrow H = p_*(\pi_1(\tilde{X}_0', \tilde{x}_0')) < \pi_1(X, x_0)$$

$$H \rightsquigarrow \tilde{X}' = H \backslash \underbrace{\tilde{X}}_{\text{universal cover}} \quad \{[\overset{\text{path in } X}{\overset{\downarrow}{\gamma}}] : \gamma(0) = x_0\}$$

Rem $\tilde{X}_1' \xrightarrow{q} \tilde{X}_2' \xrightarrow{p} X$ $q(\tilde{x}_1') = \tilde{x}_2'$ for

$(\tilde{X}_1' \xrightarrow{qp} X, \tilde{x}_1')$ & $(\tilde{X}_2' \xrightarrow{p} X, \tilde{x}_2')$ as above

gives $H_{\tilde{X}_1', \tilde{x}_1'} = \mathrm{img}\,(qp)_* < H_{\tilde{X}_2', \tilde{x}_2'}$

i.e. map of pointed coverings $\leftrightarrow$ incl. of subgrps of $\pi_1(X, x_0)$

What happens when we forget the basepoint?

$\underset{\text{path-conn.}}{\tilde{X}'} \xrightarrow{p} X$, $\tilde{x}_0$, $\tilde{x}_0' \in p^{-1}(x_0)$

$\rightsquigarrow$ we get $H = p_*(\pi_1(\tilde{X}', \tilde{x}_0))$, $H' = p_*(\pi_1(\tilde{X}', \tilde{x}_0'))$

Prop. 1. 1) $H' = gHg^{-1}$ for some $g \in \pi_1(X, x_0)$

2) any $gHg^{-1}$ is of the form $p_*(\pi_1(\tilde{X}', \tilde{x}_0''))$

for some $\tilde{x}_0'' \in p^{-1}(x_0)$

Sketch: 1) $g_0$: path from $\tilde{x}_0'$ to $\tilde{x}_0$, $g = p \circ g_0$

2) lift $\bar{g}$ to a path starting at $\tilde{x}_0$

Summary $(\underset{\text{path-conn}}{\tilde{X}'} \xrightarrow{p} X)$ classified by subgroups of $\pi_1(X, x_0)$ up to conjugacy

Rem. classification of non-connected coverings

$\tilde{X}' \to X$ s.t. $|p^{-1}(x_0)| = n$ is indep. of $x_0 \in X$

$\underset{1:1}{\longleftrightarrow}$ homomorphisms $\pi_1(X, x_0) \xrightarrow{\rho} S_n$

up to equiv. rel $\rho \sim h\rho(\sim)h^{-1}$. $h \in S_n$

sketch : $\rho(\gamma)$ is abstractly bij. $p^{-1}(x_0) \to p^{-1}(x_0)$

s.t. $\rho(\gamma)(\tilde{x}_0) = \tilde{\gamma}(1)$ , $\tilde{\gamma}$ : lift as path in $\tilde{X}'$

$\tilde{\gamma}(0) = \tilde{x}_0$

Autmorphism of covering spaces

$\tilde{X}' \xrightarrow{p} X$ covering space

Def. a <u>deck transformation</u> of $(\tilde{X}', p)$ is

a homeomorphism $f : \tilde{X}' \to \tilde{X}'$ s.t. $p = p \circ f$

Prop.2 $\tilde{X}'$ path-conn , $f, f'$ deck transf.

$\exists x' \in \tilde{X}'$ $f(x') = f'(x') \iff f = f'$

$\Leftarrow$ is obvious ; $\Rightarrow$ : take $x'' \in \tilde{X}'$

by assumption $\exists \tilde{\gamma}$ path from $x'$ to $x''$

$f \circ \tilde{\gamma}$ and $f' \circ \tilde{\gamma}$ are both paths

- lifting $\gamma = p \circ \gamma$
- starting at $f(x') = f'(x')$

$\Rightarrow$ they are same $\Rightarrow$ $\underset{\tilde{\gamma}(1)}{f(x'')} = f'(x'')$ $\square$

So $\{\text{deck transfs. of } \tilde{X}' \xrightarrow{p} X\} \to Sym(p^{-1}(x))$ (p-c. ; res.)

is inj. for any $x \in X$ (a bit stronger, inj. to orbit of $\forall \tilde{x} \in p^{-1}(x)$)

Def. $\tilde{X}' \xrightarrow{p} X$ is <u>normal</u> if $\forall x \in X$, $\tilde{x}, \tilde{x}' \in p^{-1}(x)$

$\exists$ deck transf. $f$ s.t. $f(\tilde{x}) = \tilde{x}'$

Examples 1. $X$ conn. & loc. contr. $\tilde{X}$ is normal

$(p^{-1}(x_0) = \pi_1(X_0) = \text{deck trans.})$.

2 $S^1 \to S^1$, $z \mapsto z^m$ is normal

$f^k(z) = e^{\frac{2\pi i k}{m}} z$ relate fibers of $w \in S^1$

$(z^m = w)$

Prop. 3 (1.39) $X$ path-conn. & loc. path-conn.

$\tilde{X}' \xrightarrow{p} X$ covering sp. (p-c.), $x_0 \in X$, $\tilde{x}_0 \in p^{-1}(x_0)$

$H = p_*(\pi_1(\tilde{X}', \tilde{x}_0)) < \pi_1(X, x_0) = G$

a) $(\hat{X}', p)$ normal $\Leftrightarrow$ $H$ normal in $G$

$(\forall g \in G \quad gHg^{-1} = H)$

b) $G(\tilde{X}') = \{\text{deck transf. of } \tilde{X}' \to X\}$ is

isom. to $N_G(H)/H$ $\quad N_G(H) = \{g \in G : gHg^{-1} = H\}$

Proof a) is from Prop. 1

b) Step 1 map $N_G(H) \to G(\tilde{X}')$

$g = [\gamma] \in N_G(H)$, lift $\gamma$ to a path $\tilde{\gamma}$ in

$\tilde{X}'$, $\tilde{\gamma}(0) = \tilde{x}_0$. Put $\tilde{x}_1 = \tilde{\gamma}(1)$

Then $p_*(\pi_1(\hat{X}', \hat{x}_1)) \underset{\tilde{\gamma}: \tilde{x}_0 \to \tilde{x}_1}{=} g^{-1}Hg \underset{g \in N_G(H)}{=} H$

Lifting criterion (Prop. 1.33) gives $f : \tilde{X}' \to \tilde{X}'$

s.t. $f(\tilde{x}_0) = \tilde{x}_1$

Step 2 kernel of this map is $H$

We are looking at the cond. $f(\tilde{x}_0) = \tilde{x}_0$

(Prop. 2)

i.e. $\tilde{\gamma}(1) = \tilde{x}_0$ so $[\tilde{\gamma}] \in H$. this is $g$.

Step 3 $G(\tilde{X}')$ is eq. to the img of $N_G(H)$

$f : \tilde{X}' \to \tilde{X}'$ $\rightsquigarrow$ take $\tilde{\gamma}$ connecting $\tilde{x}_0$

to $\tilde{x}_1 = f(\tilde{x}_0)$ then $g = [p \circ \tilde{\gamma}]$ gives

$f$ by Step 1 $\square$

## Cayley graph & Cayley complex

$G$ group with generating set $S$

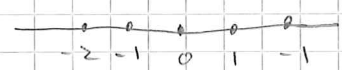
Def. <u>Cayley graph</u> of $(G, S)$ : 1-dim cell cplx $X$

$X^0 = G$ ; each $g \in G$ defs a vertex

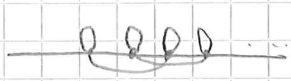
$\forall g \in G$, $s \in S$ there is an edge between

$g$ and $gs$

Ex.




$(\mathbb{Z}, \{\pm 1\})$



$\mathbb{Z}, \{0, \pm 1, \pm 2\}$

$G = \langle S \mid R \rangle$ ($R \subset F_S$).

relations

Def. <u>Cayley complex</u> of $(G, S, R)$ : 2-dim. cell

complex $X$, $X^1$ : Cayley graph of $(G, S)$

each $r = s_1 \cdots s_k \in R$, $g$ defines a 2-cell

with bdry $g \sim gs_1 \sim gs_1s_2 \sim \cdots \sim gs_1 \cdots s_k = g$

$(\mathbb{Z}^2, (x = (1,0), y = (0,1)), R = (xyx^{-1}y^{-1}))$