2021.03.09

Chapter 2 Homology

Motivation Euler characteristic

polyhedron $P \rightsquigarrow$ count # of vertices, edges, faces
V E F

$\chi(P) = |V| - |E| + |F|$

$|V|=4$, $|E|=6$, $|F|=4$; $|V|=8$
$|E|=12$, $|F|=6$

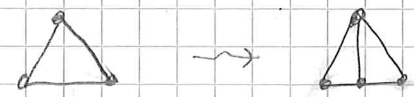
P bounds a convex dom.

$\Rightarrow \chi(P) = 2$

$\chi(P)$ only depends on the homotopy type of P.
(but not $|V|$, $|E|$, $|F|$)

$\rightsquigarrow$ want: capture the relations between
vertices (0-cells), edges (1-cells), faces (2-cells).

e.g.



should give some cancellation

basic idea

each face (w/ orientation) determine boundary edges (w/ ori.)

should cancel

same w/ edges $\rightsquigarrow$ vert. ("oriented" with $\pm$).

cancel.

§ 2.1 simplicial & singular homology

Want: "good" cell complex presentations of
top. sp to capture boundary relations.

good: polyhedra

even better: polyhedra w/ triangle faces
(and higher dim. varictions)

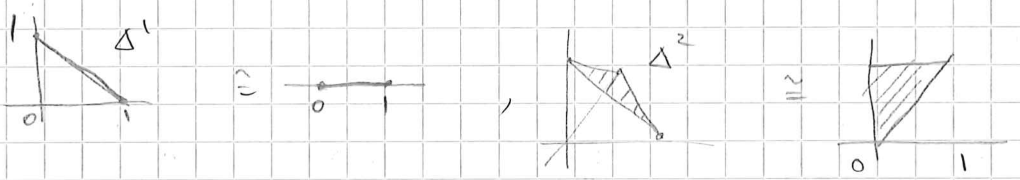
bad: S^2

one 0-cell & one 2-cell
no 1-dim part representing the boundary of 2-cell.

Def. $n = 0,1,2,\dots$; the <u>standard</u> <u>n-simplex</u> is

$$\Delta^n = \{(t_0,\dots,t_n) \in \mathbb{R}^{n+1} : \textstyle\sum t_i = 1,\ t_i \geq 0\}$$

$$\cong \{(t'_1,\dots,t'_n) \in \mathbb{R}^n : 0 \leq \underset{t_0}{\underset{\shortparallel}{t'_1}} \leq \underset{t_0+t_1}{\underset{\shortparallel}{t'_2}} \leq \dots \leq \underset{t_0+\dots+t_{n-1}}{\underset{\shortparallel}{t'_n}} \leq 1\}$$



More generally an <u>n-simplex</u> is an n-dim.

convex subset of $\mathbb{R}^m$ with $(n+1)$ extremal points
(together w/ ordering)

i.e. $[v_0,\dots,v_n]$ representing

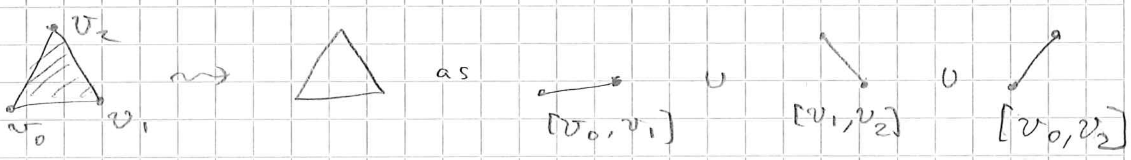
$$\{\sum_{i=0}^n t_i v_i : t_i \in \Delta^n\} \subset \mathbb{R}^m$$

$v_1 - v_0, \dots, v_n - v_0$ lin. indep.

So $\Delta^n = [v_0,\dots,v_n]$ with $v_i = (0,\dots,\underset{i\text{-th}}{1},\dots,0)$.

Def $[v_0,\dots,v_n]$ n-simplex; its <u>face</u> is an $(n-1)$-simplex of the form $[v_0,\dots,\underset{\text{omit.}}{\hat{v}_i},\dots,v_n]$

<u>boundary</u> of n-simplex : union of its faces



(for the moment ignore ordering)

Rem interpret ordering of vertices as <u>orientation</u> and swapping two verts. as opposite orientation

$$[v_0,v_1,v_2] \xleftrightarrow{\text{oppos.}} [v_2,v_1,v_0],\ [v_1,v_0,v_2]$$
$$[v_2,v_0,v_1],\ [v_1,v_2,v_0] \qquad [v_0,v_2,v_1]$$

Notation $\partial\Delta^n$ boundary

$\mathring{\Delta}^n = \Delta^n \setminus \partial\Delta^n$ interior.

$\mathring{\Delta}^0 = \Delta^0 = \{1\} \subset \mathbb{R}$

Def. a Δ-complex is given by

- top. sp. X ("underlying top. sp.")
- maps $\sigma_\alpha : \Delta^n \to X$ (cont.) $\alpha \in I_n$ for some ind. set

s.t.

1.) $\sigma_\alpha|_{\mathring{\Delta}^n}$ is injective ;

$$X = \coprod_{n=0}^{\infty} \coprod_{\alpha \in I_n} \sigma_\alpha(\mathring{\Delta}^n) \quad \text{(disj. union)}$$

2) $\forall \alpha \in I_n$, $\forall$ face $f_i = [v_0, \dots, \hat{v}_i, \dots, v_n]$ of Δ^n

$\exists!\ \beta \in I_{n-1}$ s.t. $\sigma_\alpha|_{f_i} \circ \varphi = \sigma_\beta$

$\varphi : \Delta^{n-1} \to f_i$ given by

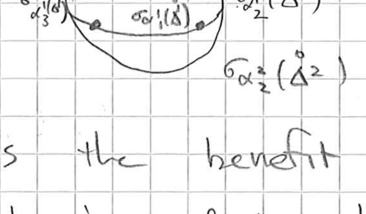
$$(t_0, \dots, t_{n-1}) \mapsto (t_0, \dots, \underset{i\text{th}}{0}, \dots, t_{n-1})$$

$$\sum t_j v_j \qquad \sum_{j<i} t_j v_j + \sum_{j \geq i} t_j v_{j+1}$$

3) $U \subset X$ is open $\Leftrightarrow$ $\sigma_\alpha^{-1}(U) \subset \Delta^n$ open for all n, $\alpha \in I_n$

(weak top. defied by $\sigma_\alpha : \Delta^n \to X$, $\alpha \in I_n, n$)

Rem. a Δ-complex can be regarded as a cell complex up to homeo. $D^n \cong \Delta^n$.



(but not conversely in gen.)

What's the benefit ?

glueing data becomes completely combinatorial

enough to remember :

- $\forall n$: index sets I_n for n-cells / simplices in X
- $\forall n$, $0 \leq i \leq n$: "face" map : $d_i^n : I_n \to I_{n-1}$

with consistency conditions $d_i^{n-1} d_j^n = d_{j-1}^{n-1} d_i^n$ $(i < j)$

(semisimplicial set)

simplicial homology

$(X, \{I_n\}_n, \partial_i^n : I_n \to I_{n-1})$ Δ-complex
"i-th face"

Def. $\underline{n}$-chain group of X : $\Delta_n(X) = \mathbb{Z} I_n$

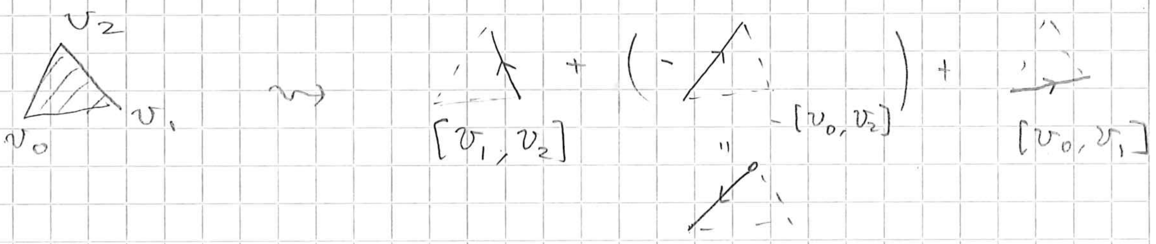
$\cong \{$formal lin. comb. $\sum_{\alpha \in I_n} n_\alpha \sigma_\alpha$: $n_\alpha \in \mathbb{Z}\}$

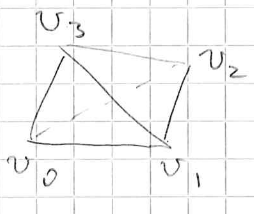
or $\sum_{\alpha \in I_n} n_\alpha e_\alpha^n$ $\quad e_\alpha^n = \sigma_\alpha(\mathring{\Delta}^n)$.

the boundary map $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$

linear ext. of $\partial_n(\sigma_\alpha) = \sum_{i=0}^n (-1)^i \underbrace{\sigma_\alpha|_{[v_0, \dots, \hat{v}_i, \dots v_n]}}_{\sigma_\beta \text{ for } \exists! \beta \in I_{n-1} \text{ by } \Delta\text{-cplx structure}}$

Rem the sign $(-1)^i$ takes care of orientation





$[v_0, v_3, v_2]$ would have "same" orientation as $[v_1, v_2, v_3]$, ...

Prop (Lem 2.1) $\partial_{n-1} \partial_n = 0$

Proof apply ∂_{n-1} to $\partial_n(\sigma_\alpha) = \sum_{i=0}^n (-1)^i \sigma_\alpha|_{(v_0, \dots, \hat{v}_i, \dots v_n]}$

$\rightsquigarrow \sum_{j=0}^{n-1} \sum_{i=0}^{n} (-1)^{j+i} \sigma_\alpha|_{\text{omit } j\text{-th vert of } [v_0, \dots \hat{v}_i, \dots v_n]}$

$= \sum_{j<i} (-1)^{j+i} \sigma_\alpha|_{[\dots \hat{v}_j \dots \hat{v}_i \dots]}$

$+ \underbrace{\sum_{j \geq i} (-1)^{j+i} \sigma_\alpha|_{[\dots \hat{v}_i \dots \hat{v}_{j+1} \dots]}}_{\sum_{j' > i} (-1)^{i+j'-1} \sigma_\alpha|_{[\dots \hat{v}_i \dots \hat{v}_{j'} \dots]}} = 0$