2021.03.10 no. 1

Problem set 4 (§ 1.3)

Exercise 9

X : path-conn., loc. path-conn., $\pi_1(X, x)$ finite.

Show : any cont. map $X \xrightarrow{f} S^1$ is homotopic to a const. map.

Step 1 f lifts to a cont. map $\tilde{f} : X \to \mathbb{R}$.

(for $\mathbb{R} \xrightarrow{p} S^1$ std. proj.)

Use lifting criterion (Prop 1.33).

$p : \tilde{Z}' \to Z$ covering $p(\tilde{z}_0) = z_0$

$f : X \to Z$ cont. $f(x_0) = z_0$

X path-conn., loc. path-conn.

then $\exists\ \tilde{f} : X \to \tilde{Z}$ lift of f, $\tilde{f}(x_0) = \tilde{z}_0$

$\Leftrightarrow f_*(\pi_1(X, x_0)) \subset p_*(\pi_1(\tilde{Z}', \tilde{z}_0))$

Now we have $\tilde{Z}' = \mathbb{R}$ so $p_*(\pi_1(\tilde{Z}', \tilde{z}_0)) = \{e\}$.

we need to show $f_*(\pi_1(X, x_0)) = \{e\}$

- $f_* \pi_1(X, x_0) < \pi_1(S^1, f(x_0)) \simeq \mathbb{Z}$.

- $\pi_1(X, x_0)$ finite $\Rightarrow$ so is $f_*(\pi_1(\cdots))$

$\{0\}$ is the only fin. subgroup of $\mathbb{Z}$.
(e in $\mathbb{Z}$)

Step 2. $f \simeq$ const map.

$\tilde{f}$ is homotopic to a const map.

(we can take $F(x, t) = t\tilde{f}(x)$

as $\tilde{f} \simeq$ const. to 0)

$\Rightarrow p \circ \tilde{f} = f$ is homotopic to const map.

Exercise 14

Find all connected covering spaces of $\mathbb{R}P^1 \vee \mathbb{R}P^1$

Recall (March 3) $\left(\underset{\text{path-conn.}}{\tilde{X}'} \xrightarrow{p} X \ : \text{cov. sp.}\right)$ are classified by subgroups of $\pi_1(X, x_0)$ up to conjungacy

Step 1 computation of $\pi_1(\mathbb{R}P^2, *)$

$\mathbb{R}P^2 \cong S^2 / x \sim (-x)$. (as $S^2 \subset \mathbb{R}^3$).

So we have covering $S^2 \to \mathbb{R}P^2$

- S^2 simply conn. $\Rightarrow$ this is a model of univ. cover $\widetilde{\mathbb{R}P^2}$

- $X \cong \pi_1(X, x_0) \backslash \tilde{X}$ for conn. loc. contr. X.

$\Rightarrow \pi_1(\mathbb{R}P^2, *) \cong \{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$ as group

Step 2 $\pi_1(\mathbb{R}P^1 \vee \mathbb{R}P^1, *) \cong \underset{\text{semidirect prod.}}{\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}}$

Van Kampen's thm gives $\pi_1(\mathbb{R}P^2 \vee \mathbb{R}P^2, *) \cong \pi_1(\mathbb{R}P^2, *) * \pi_1(\mathbb{R}P^2, *)$

$\underset{\text{Step 1}}{\cong} (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$

Recall (Feb. 10) $\underset{\langle a \mid a^2 = e\rangle}{(\mathbb{Z}/2\mathbb{Z})} * \underset{\langle b \mid b^2 = e\rangle}{(\mathbb{Z}/2\mathbb{Z})} \cong \mathbb{Z} \rtimes \underbrace{(\mathbb{Z}/2\mathbb{Z})}_{\{e, g\}}$

$a \longmapsto (1, g)$

$b \longmapsto (0, g)$

Step 3 subgroups of $\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$

- $m\mathbb{Z} < \mathbb{Z}$ for $m = 0, 1, 2, \cdots$ mutually nonconj.

- if $H < \mathbb{Z} \rtimes (\mathbb{Z}/2\mathbb{Z})$ contains $h \notin \mathbb{Z}$

  then $\exists\ h' \in \mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ s.t. $h' H h'^{-1} \ni (0, g)$ or $(1, g)$.

  $h = (n, g)$ is conj to $(n + 2k, g)$ by (k, e)

(conceptually : img in $\{e\} * (\mathbb{Z}/2\mathbb{Z})$, $(\mathbb{Z}/2\mathbb{Z}) * \{e\}$.)

$\rightsquigarrow$ $2k\mathbb{Z} \vee \langle(0,g)\rangle$, $2k\mathbb{Z} \vee \langle(1,g)\rangle$.

$(2k+1)\mathbb{Z} \vee \langle(0,g)\rangle \sim (2k+1)\mathbb{Z} \vee \langle(1,g)\rangle$.

for $k = 0, 1, 2, \ldots$

Exercise 23

$G \curvearrowright X$ freely & properly discontinuous

$\uparrow$ Hausdorff

$\rightsquigarrow$ $X \to G\backslash X$ is covering (and $G \curvearrowright X$ becomes a deck transformation action)

condition on the action : $\forall x \in X$ $\exists \underset{\text{open}}{U} \ni x$ s.t.

$\{g \in G : U \cap g(U) \neq \emptyset\}$ is finite , $gx = x \Rightarrow g = e$

topology on $G\backslash X = \{[x] : x \in X, \ [gx] = [x] \text{ for } g \in G\}$

$V \subset G\backslash X$ is open iff it's the img of a G-inv. open set $\widetilde{U} \subset X$.

Want : given $x \in X$ find small $U_1 \ni x$ s.t. gU_1 $(g \in G)$ are mutually disjoint

and we'll check that the img of GU is an evenly covered neigh of $[x]$.

(want $y \in U$, $gy \in U$

Step 1 $\exists U_1 \ni x$ open neighborhood s.t.

$gU_1 \cap U_1 \neq \emptyset \iff g_1 = e$

Take U as in properness ass.

$\{g_1, \ldots, g_n\} = \{g \in G : gU \cap U \neq \emptyset\}$

take ${}_{x\in}U_1 \subset U$ s.t. $g_i \neq e \Rightarrow U_1 \cap g_i U_1 = \emptyset$

(use Hausdoff-ness of X to separate

x from $g_i x \neq x$)

checking $gy \in U_1$ for some $y \in U_1 \to g = e$

$gU \cap U$ would contain $gy \Rightarrow g = g_i$ for some i

U_1 separates x from $g_i x$ for $g_i \neq e$

so $g_i x = x$ i.e. $g = e$

Step 2 any G-inv. $W \underset{\text{open}}{\subset} GU_1$ is of the

form GW_1 for uniquely determined $W_1 \underset{\text{open}}{\subset} U_1$

$W_1 = W \cap U_1$ works.

Step 1.5 $gU_1 \cap g'U_1 \neq \emptyset \Leftrightarrow g = g'$

i.e. $GU_1 = \coprod_{g\in G} gU_1$

so open sets of the img of GU_1

$\leftrightarrow$ open sets of U_1

$\leftrightarrow$ open sets of gU_1 for any fixed $g \in G$

i.e. $V = [GU_1]$ is an evenly covered

neigh. of $[x]$.